Generalization of the Coddington Equations to Include Hybrid Diffractive Surfaces

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ABSTRACT

Coddington Equations are used to calculate the astigmatic images of a small bundle of rays centered on a ray commonly known as the principal ray. Some authors generalize it such that for a refractive or reflective surface of any shape to the 2^{nd} order, and an incident wavefront of any shape to the 2^{nd} order, the refracted or reflected wavefront can be calculated to the 2^{nd} order. We extend it further such that it applies to the diffractive surface as well. The derivation is based on the general Snell's law and differential ray tracing approach. We present these generalized Coddington Equations in two forms: matrix formalism and explicit expressions. The equations are verified with explicit ray tracing using a commercial lens design program.. The relations are applied to evaluate the imaging performance for null testing of aspheric surfaces using computer generated holograms.

Keywords: Interferometric imaging; Testing; Aberration; Optical design.

1. INTRODUCTION

Coddington's equations were first derived to find out the astigmatism and field curvature of the image of a point through a spherical surface[1]. Over the years, people generalized Coddington equations such that they can be used to calculate the astigmatism and curvature in the refracted or reflected wavefront when the same characteristics of the incident wavefront and the refractive or reflective surface are given[2-5]. In some cases, diffractive surfaces are also part of the system where we need to find out the image locations; one example is the null test system for an aspheric surface where a CGH is the null lens or part of the null lens. So there is a need to further generalize the Coddington equations that deal with the diffractive surface as well. In this paper, we apply the generalized Snell's law and use the differential ray tracing approach to derive the more generalized Coddington equations for refractive/diffractive hybrid surfaces. Section 2 describes the basic derivation steps and gives the Equations in two formats: matrix form and explicit expressions. The derived equations were verified with a general model set up in Zemax. Details of the model are given in Section 3. An example application of the new equations, calculation of the image locations of the test surface through the CGH in an interferometric null test system, is given in Section 4.

2. DERIVATION

A diffractive surface can be modeled as a phase object that is described with a phase function $\Phi(x, y)$. The gradient of the phase function determines the diffracted ray direction as shown in the general Snell's law:

$$(n_1\hat{r}_1 - n_2\hat{r}_2 + \nabla\Phi)\times\hat{n} = 0\,, \tag{1}$$

where $\hat{r}_1$ and $\hat{r}_2$ are the unit vectors along the incident and outgoing principal rays, respectively, and $\hat{n}$ is the unit vector along the surface normal[6].

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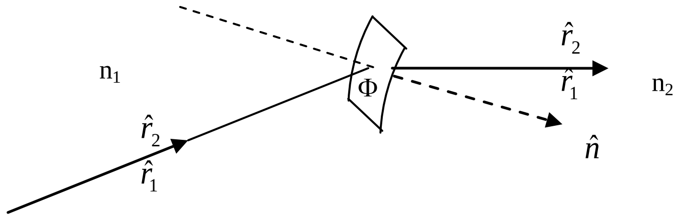


Figure 1. Illustration of a ray incident on a refractive and diffractive hybrid surface getting bent after the surface.

The second order terms of the phase, which represent the curvatures, determine the power and astigmatism of the refracted/diffracted wavefront. Our derivation is intended to solve the following problem: given the curvatures of the incident wavefront, the refractive or reflective surface and the diffractive phase function Φ, find out the power and astigmatism of the refracted/diffracted wavefront.

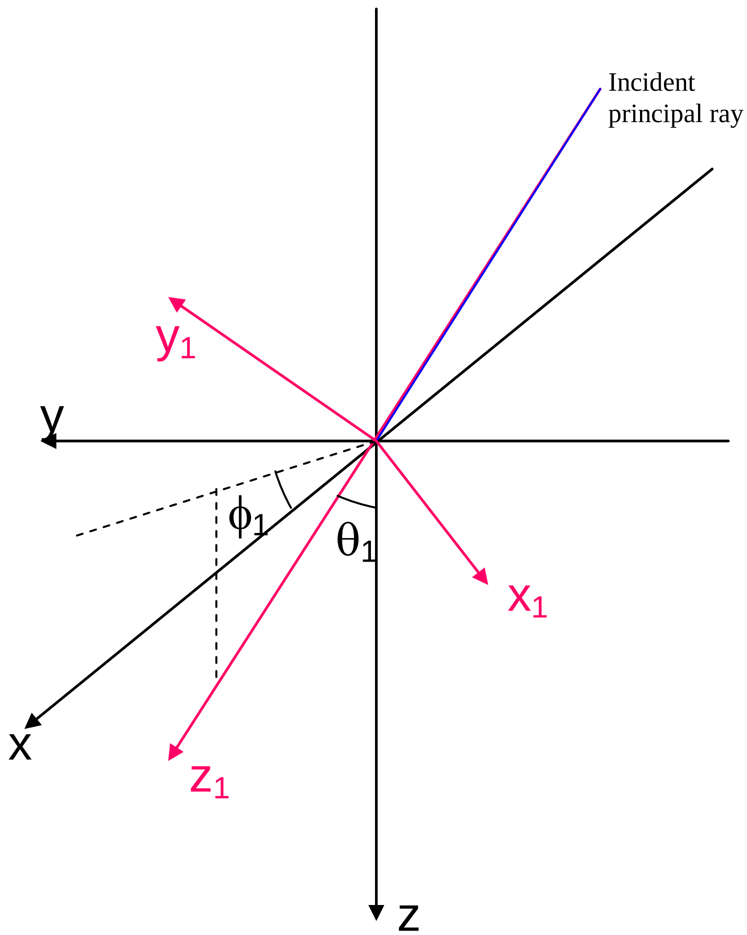


Figure 2. Illustration of coordinate systems' definitions for the refractive/reflective and diffractive surface and the incident wavefront: x-y-z is the coordinate for the surface and x1-y1-z1 is for the incident wavefront . The coordinate system for the refracted/reflected/diffracted wavefront is defined similarly as that of the incident wavefront.

The first step to attack the problem is to define all the curvatures. To do that we need to establish coordinate systems for all three spaces involved: the incident wavefront, the surface, and the refracted/diffracted or reflected/diffracted wavefront. When a ray is refracted or reflected at a surface, the incident ray, the surface normal and the outgoing ray lie in the same plane. This makes the choice of the coordinate system simple – one of the axes is chosen to be perpendicular to the plane in each space: the incident media, the surface and the following media. Then the resultant Coddington equations take a simple form. For the refractive/diffractive or reflective/diffractive hybrid surface, normally the incident ray, the surface normal and the refracted/diffracted or reflected/diffracted ray do not lie in the same plane; therefore, there is no obvious choice of coordinate systems in the three spaces. We define the coordinate systems as follows:

1. The z-axis of the refractive/diffractive or reflective/diffractive surface is along the normal of the surface pointing to medium 2. The x and y axes are arbitrarily chosen.
2. The z-axes of the incident and refracted/diffracted or reflected/diffracted wavefronts are along the principal rays. The principal ray vectors are $\mathbf{z_1}$=($\sin\theta_1\cos\phi_1$, $\sin\theta_1\sin\phi_1$, $\cos\theta_1$) and $\mathbf{z_2}$=($\sin\theta_2\cos\phi_2$, $\sin\theta_2\sin\phi_2$, $\cos\theta_2$), respectively, in the surface coordinate frame.
3. The x-axis of the incident wavefront is chosen to be perpendicular to the incidence plane, i.e. $\mathbf{x_1}=\mathbf{z_1}\mathrm{x}\mathbf{z}$, then $\mathbf{y_1}$ is determined following the right-hand rule. The x-y axes of the refracted/diffracted or reflected/diffracted wavefront are chosen similarly, i.e. $\mathbf{x_2}=\mathbf{z_2}\mathrm{x}\mathbf{z}$, and $\mathbf{y_2}=\mathbf{z_2}\mathrm{x}\mathbf{x_2}$.

For a function S(x,y), we define a curvature matrix as

$$\begin{bmatrix} C_{xx} & C_{xy} \\ C_{xy} & C_{yy} \end{bmatrix} = \begin{bmatrix} \partial^2 / \partial x^2 & \partial^2 / \partial x \partial y \\ \partial^2 / \partial x \partial y & \partial^2 / \partial y^2 \end{bmatrix} S(x,y). \qquad (2)$$

For the incident wavefront, and the surface and the diffractive phase function, their curvature matrices are given as $\begin{bmatrix} C_{xx1} & C_{xy1} \\ C_{xy1} & C_{yy1} \end{bmatrix}$, $\begin{bmatrix} C_{xxr} & C_{xyr} \\ C_{xyr} & C_{yyr} \end{bmatrix}$ and $\begin{bmatrix} C_{xxd} & C_{xyd} \\ C_{xyd} & C_{yyd} \end{bmatrix}$. We want to find out the curvature matrix of the refracted/diffracted or reflected/diffracted wavefront: $\begin{bmatrix} C_{xx2} & C_{xy2} \\ C_{xy2} & C_{yy2} \end{bmatrix}$.

We first apply Snell's law to find the refracted/diffracted ray direction. To see the how the rays around the principal ray are changed due to refracted or reflectred diffraction , we differentiate Eq. (1) and obtain

$$(n_1 d\hat{r}_1 - n_2 d\hat{r}_2 + d\nabla\Phi) \times \hat{n} + (n_1 \hat{r}_1 - n_2 \hat{r}_2 + \nabla\Phi) \times d\hat{n} = 0. \qquad (3)$$

For an arbitrary ray around the incident principal ray with known $d\hat{r}_1$, we would know where it hits the surface given the curvature matrix of the incident wavefront. Then we can calculate $d\nabla\Phi$ and $d\hat{n}$, again by using the given curvature matrices of the refractive surface and the phase function. Solving Eq. (3) gives the refracted/diffracted ray direction $d\hat{r}_2$. The curvature matrix of the refracted/diffracted wavefront follows. The result is shown below. The detailed derivation is given in Reference 7.

The matrix form of the derived Coddington Equations is:

$$
\begin{aligned}
&n_2\begin{bmatrix} \sin\phi_2 & \cos\phi_2\cos\theta_2 \\ -\cos\phi_2 & \sin\phi_2\cos\theta_2 \end{bmatrix}\cdot\begin{bmatrix} C_{xx2} & C_{xy2} \\ C_{xy2} & C_{yy2} \end{bmatrix}\cdot\begin{bmatrix} \sin\phi_2 & -\cos\phi_2 \\ \cos\phi_2\cos\theta_2 & \sin\phi_2\cos\theta_2 \end{bmatrix}- \\
&n_1\begin{bmatrix} \sin\phi_1 & \cos\phi_1\cos\theta_1 \\ -\cos\phi_1 & \sin\phi_1\cos\theta_1 \end{bmatrix}\cdot\begin{bmatrix} C_{xx1} & C_{xy1} \\ C_{xy1} & C_{yy1} \end{bmatrix}\cdot\begin{bmatrix} \sin\phi_1 & -\cos\phi_1 \\ \cos\phi_1\cos\theta_1 & \sin\phi_1\cos\theta_1 \end{bmatrix} \\
&=(n_2\cos\theta_2-n_1\cos\theta_1)\begin{bmatrix} C_{xxr} & C_{xyr} \\ C_{xyr} & C_{yyr} \end{bmatrix}+\begin{bmatrix} C_{xxd} & C_{xyd} \\ C_{xyd} & C_{yyd} \end{bmatrix}
\end{aligned}
\quad . \qquad (4)
$$

And the explicit expressions are:

$$
\begin{aligned}
&n_2(C_{xx2}\sin^2\phi_2+C_{xy2}\sin 2\phi_2\cos\theta_2+C_{yy2}\cos^2\phi_2\cos^2\theta_2)-n_1(C_{xx1}\sin^2\phi_1+C_{xy1}\sin 2\phi_1\cos\theta_1+C_{yy1}\cos^2\phi_1\cos^2\theta_1) \\
&=(n_2\cos\theta_2-n_1\cos\theta_1)C_{xxr}+C_{xxd}
\end{aligned}
,
$$

$$
\begin{aligned}
&n_2(C_{xx2}\cos^2\phi_2-C_{xy2}\sin 2\phi_2\cos\theta_2+C_{yy2}\sin^2\phi_2\cos^2\theta_2)-n_1(C_{xx1}\cos^2\phi_1-C_{xy1}\sin 2\phi_1\cos\theta_1+C_{yy1}\sin^2\phi_1\cos^2\theta_1) \\
&=(n_2\cos\theta_2-n_1\cos\theta_1)C_{yyr}+C_{yyd}
\end{aligned}
,
$$

$$
\begin{aligned}
&n_2(-1/2\cdot C_{xx2}\sin 2\phi_2-C_{xy2}\cos 2\phi_2\cos\theta_2+1/2\cdot C_{yy2}\sin 2\phi_2\cos^2\theta_2)-n_1(-1/2\cdot C_{xx1}\sin^2\phi_1- \\
&C_{xy1}\cos 2\phi_1\cos\theta_1+1/2\cdot C_{yy1}\sin 2\phi_1\cos^2\theta_1) \\
&=(n_2\cos\theta_2-n_1\cos\theta_1)C_{xyr}+C_{xyd}
\end{aligned}
\quad .
$$

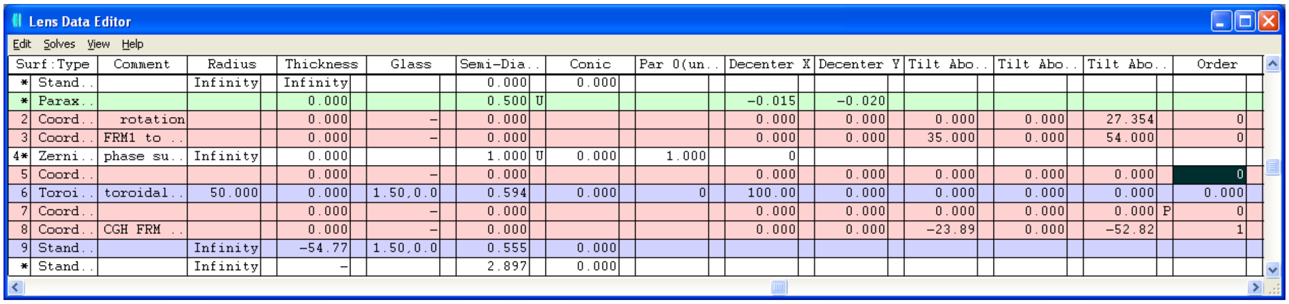
3. VERIFICATION:

To verify that the equations are correct, we set up a general model in Zemax[8] to test it. The model consists of the following surfaces in order:

1. A surface to create an arbitrary amount of astigmatism at fixed orientations
2. A surface to rotate the astigmatism an arbitrary angle
3. A surface to set an arbitrary incidence angle of the principal ray
4. A refractive toroidal surface whose curvatures can be set at any values with variable orientation
5. A diffractive surface on top of the refractive surface whose curvatures can be set at any values
6. A surface to align the image space to the coordinate system of the refracted/diffracted rays

A Zemax macro is written to calculate the angles of the refracted/diffracted principal ray which are then fed to Surface 6. The macro calculates the curvature matrices of the incident wavefront, of the refractive or reflective surface and of the diffractive phase function, and saves them into a text file. A separate Matlab program was written to implement the generalized Coddington Equations. It reads in the text file generated by the Zemax macro and calculates the curvature matrix of the outgoing wavefront and then translate it into the astigmatic image locations and angles of the principal axes. We evaluate these numbers in Zemax model and verify the astigmatic images locations and angles.

Figure 3 shows the model in Zemax. For a particular example, the generalized Coddington Equations predict the images at -54.77mm and -7.62mm, respectively and astigmatism is along angle of 31.15°. Figure 4 shows the two astigmatic images at these two locations. The angle of the line images matches the Coddington Equations' prediction as well.



Lens Data Editor

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* Stand..		Infinity	Infinity		0.000	0.000							
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5 Coord..			0.000	-	0.000			0.000	0.000	0.000	0.000	0.000	0
6 Toroi..	toroidal..	50.000	0.000	1.50,0.0	0.594	0.000	0	100.00	0.000	0.000	0.000	0.000	0.000
7 Coord..			0.000	-	0.000			0.000	0.000	0.000	0.000	0.000 P	0
8 Coord..	CGH FRM ..		0.000	-	0.000			0.000	0.000	-23.89	0.000	-52.82	1
9 Stand..		Infinity	-54.77	1.50,0.0	0.555	0.000							
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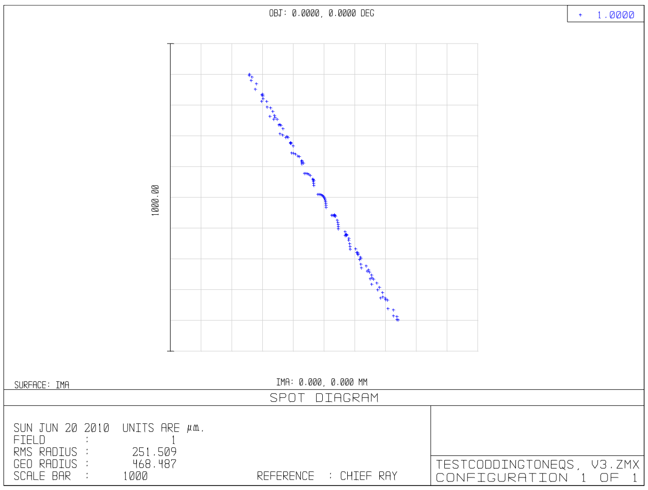
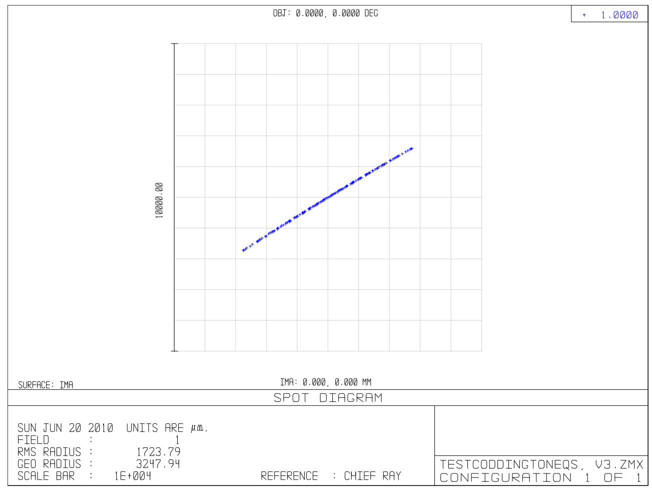
Figure 3. A design example for the general model set up in Zemax to verify the generalized Coddington Equations.



Spot diagram at 7.62 mm



Spot diagram at 54.77mm.

Figure 4. Explicit Zemax simulation showing the spot diagrams for image locations predicted by the generalized Coddington Equations to provide principal astigmatic focus.

4. APPLICATION

The computer generated hologram (CGH), a diffractive element, is often used as the null lens or part of the null lens in optical testing. It forms intermediate images of the test surface; these images are then relayed to the camera by the interferometer optics. Aberrations of the intermediate images, primarily astigmatism and field curvatures, affect measurements of detailed features on the test surface[9]. The Generalized Coddington Equations we derived above are the tools for analyzing the intermediate image quality.[10]

Figure 5 shows the interferometric null test system of the Giant Magellen Telescope's primary segment[11]. The null lens consists of a 3.75m diameter large fold sphere, a 0.75m diameter small fold sphere and a CGH. The null lens forms astigmatic images of the GMT segment under test. In the Zemax model, we traced rays originating from the GMT segment and perpendicular to segment surface through the null lens. We recorded the incident/outgoing ray directions of the rays at each and every surface, as well as the curvature matrix at the CGH surface. Then we used the generalized Coddington Equations to calculate the astigmatic images' location and orientation. This information is then used to determine the measurement transfer function.

The above mentioned work can be done by tracing a bundle of rays from a point on the test surface. Yet the equations give us the insight and allow us to create a parametric model for designing the CGH that satisfies the imaging requirement of the test system.[12]

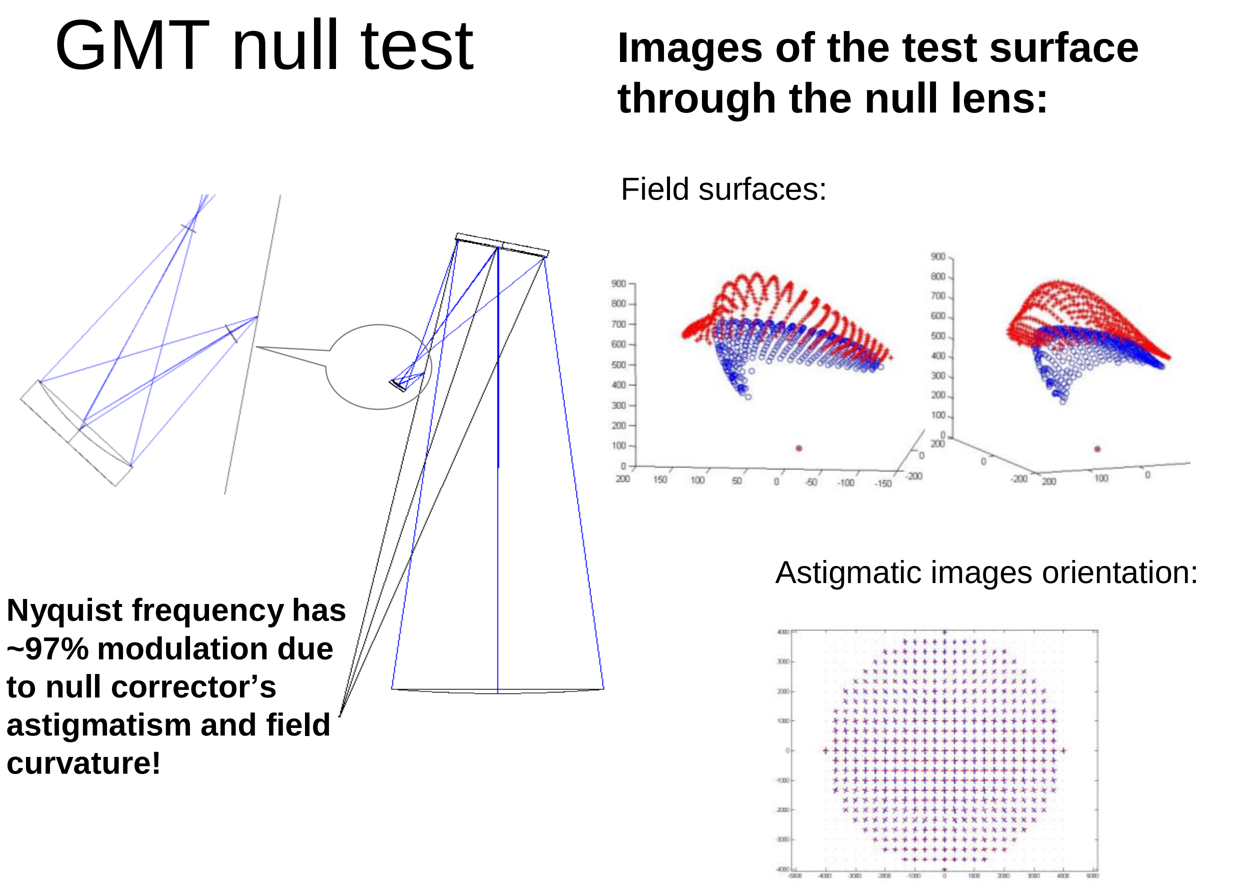


Figure 5. Illustration of the GMT null test system, and the locations and orientation of the astigmatic images of the test surface formed by the null optics.

5. DISCUSSION

It has been publicly intimated that the relationships presented here had been incorporated in the CodeV[13] optical design code for many years. While CodeV may indeed determine the field curves correctly by tracing rays through the system, there is no evidence that indicates prior derivation or use of the generalized Coddington Equations.

6. REFERENCES

1. R. Kingslake, "Who discovered Coddington's equations?" Opt. Photon. News May 1994, pp. 20–23.
2. O. N. Stavroudis, ‘‘A simpler derivation of the formulas of generalized ray tracing,’’ J. Opt. Soc. Am. **66**, 1330–1333 (1976).
3. D. G. Burkhard and D. L. Shealy, "Simplified formula for the illuminance in an optical system," Appl. Opt. **20,** 897–909 (1981).
4. Charles E. Campbell, "Generalized Coddington equation in ophthalmic lens design," J. Opt. Soc. Am. A **23**, 1691– 1698 (2006).
5. J. E. A. Landgrave and Jesu´ s R. Moya-Cessa, "Generalized Coddington equations in ophthalmic lens design," J. Opt. Soc. Am. A **13**, 1637– 1644 (1996).
6. See Eq. (5) at http://www.optics.arizona.edu/opti689/Opti689 Lecture 02.pdf

7. C. Zhao and J. H. Burge, "Generalized Coddington Equations," to be submitted to Optics Express (2010).
8. Zemax is an optical design code of Zemax Development Corporation (www.zemax.com).
9. C. Zhao and J. H. Burge, "Imaging aberrations from null correctors," Proc. SPIE, Vol. 6723 67230L.
10. J. H. Burge, P. Zhou, and C. Zhao, "Imaging issues for interferometry using CGH null correctors," Proc. SPIE vol. 7739 (2010)
11. J. H. Burge, et al, "Design and analysis for interferometric measurements of the GMT primary mirror segments," Proc. SPIE 6273-22.
12. P. Zhou, J. H. Burge, and C, Zhao, "Imaging issues for interferometric measurement of aspheric surfaces using CGH null correctors," Proc. SPIE Vol. 7790 (2010).
13. CodeV is an optical design code of Optical Research Associates (www.opticalres.com).