Coddington Equations

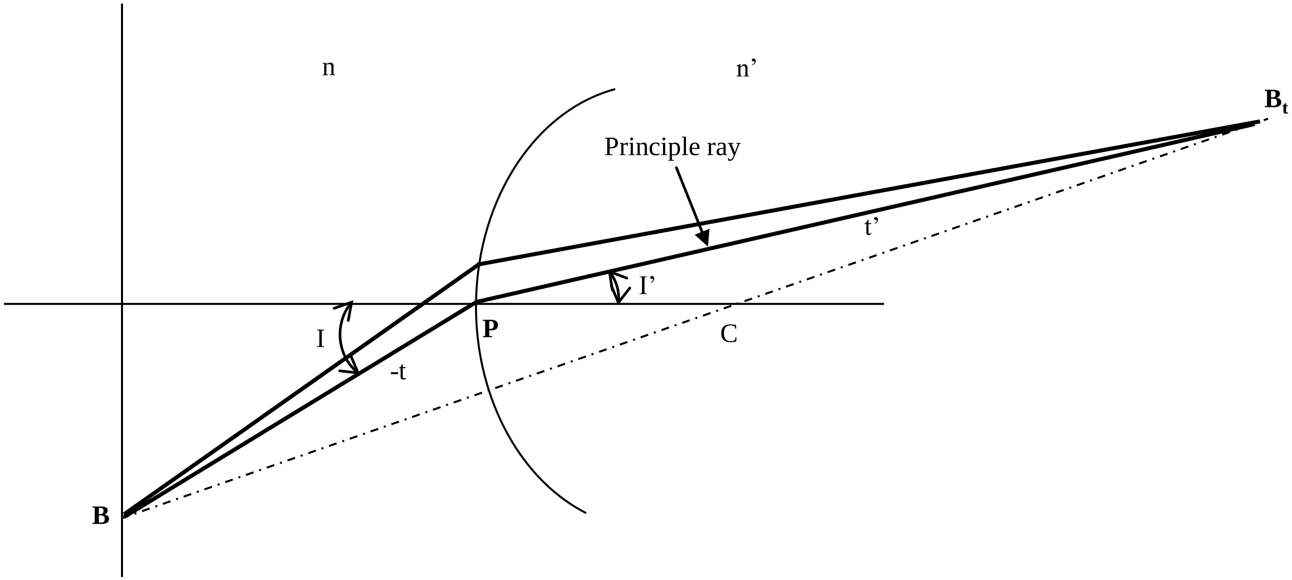
Chunyu Zhao
James H. Burge

College of Optical Sciences, University of Arizona

Outline

- Coddington Equations in the simplest form
- Coddington Equations – generalized
- Coddington Equations – further generalized
  - Derivation
  - Verification
  - Application

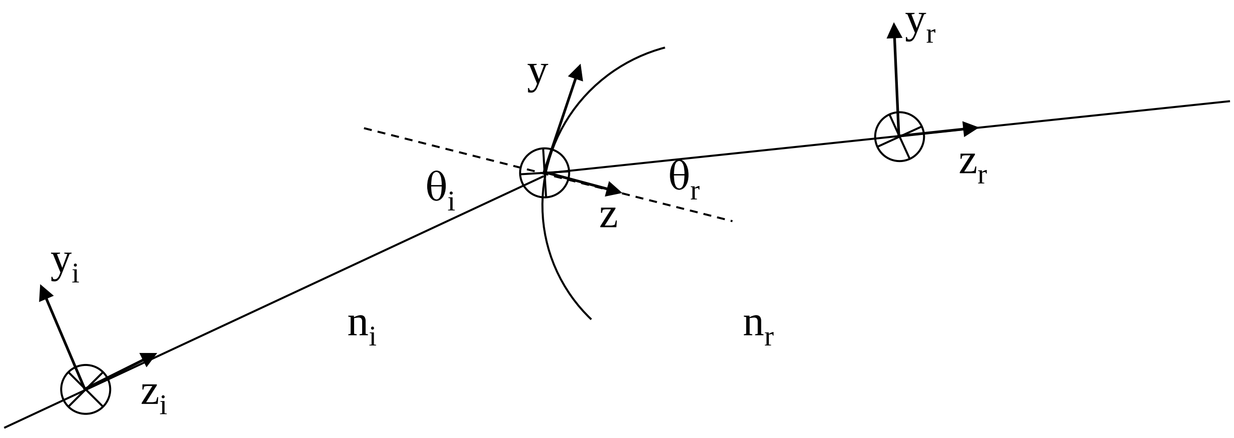
Coddington equations



$$\frac{n'}{s'} - \frac{n}{s} = \frac{n'\cos I' - n\cos I}{R}$$

$$\frac{n'\cos^2 I'}{t'} - \frac{n\cos^2 I}{t} = \frac{n'\cos I' - n\cos I}{R}$$

Coddington Equations – generalized



Wavefront/surface shape:

$$W_i = \frac{x^2}{2R_{ix}} + \frac{xy}{R_{ixy}} + \frac{y^2}{2R_{iy}}$$

$$S = \frac{x^2}{2R_x} + \frac{xy}{R_{xy}} + \frac{y^2}{2R_y}$$

$$W_r = \frac{x^2}{2R_{rx}} + \frac{xy}{R_{rxy}} + \frac{y^2}{2R_{ry}}$$

Coddington Equations – generalized:

$$\boxed{\frac{n_r}{R_{rx}} - \frac{n_i}{R_{ix}} = \frac{n_r \cos\alpha_r - n_i \cos\alpha_i}{R_x}}$$

$$\boxed{\frac{n_r \cos\alpha_r}{R_{rxy}} - \frac{n_i \cos\alpha_i}{R_{ixy}} = \frac{n_r \cos\alpha_r - n_i \cos\alpha_i}{R_{xy}}}$$

$$\boxed{\frac{n_r \cos^2\alpha_r}{R_{ry}} - \frac{n_i \cos^2\alpha_i}{R_{iy}} = \frac{n_r \cos\alpha_r - n_i \cos\alpha_i}{R_y}}$$

Curvature matrix

- Define curvature matrix for a surface or wavefront or phase function of quadratic shape

Surface or wavefront or phase shape: $$S = \frac{x^2}{2R_x} + \frac{xy}{R_{xy}} + \frac{y^2}{2R_y}$$

Curvature matrix:

$$C = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{bmatrix} S = \begin{bmatrix} \frac{1}{R_x} & \frac{1}{R_{xy}} \\ \frac{1}{R_{xy}} & \frac{1}{R_y} \end{bmatrix} = \begin{bmatrix} C_{xx} & C_{xy} \\ C_{xy} & C_{yy} \end{bmatrix}$$

Matrix Form

$$
n_r \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta_r \end{bmatrix} \cdot \begin{bmatrix} C_{xxr} & C_{xyr} \\ C_{xyr} & C_{yyr} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta_r \end{bmatrix} -
$$

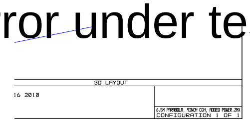
$$
n_i \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta_i \end{bmatrix} \cdot \begin{bmatrix} C_{xxi} & C_{xyi} \\ C_{xyi} & C_{yyi} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta_i \end{bmatrix}
$$

$$
= (n_r \cos\theta_r - n_i \cos\theta_i) \begin{bmatrix} C_{xx} & C_{xy} \\ C_{xy} & C_{yy} \end{bmatrix}
$$

How general are they?

- They work for both refractive and reflective surface.
- They work for incident wavefront of any shape and the surface of any shape
- A diffractive surface forms images too!
  - Fresnel zone plate
  - Computer Generated Holograms (CGH) in an interferometric null test system

Mirror under test

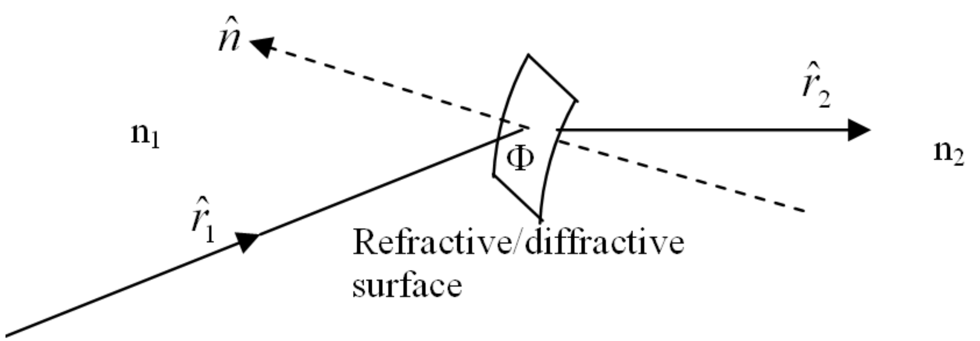


CGH

Interferometer focus

Diffractive surface

- It is represented by a phase function, Φ.
- Ray tracing through a diffractive surface using the general Snell's Law
- Incident ray, surface normal and outgoing (refracted/reflected/diffracted) ray are NOT normally in one plane, therefore, there is no convenient choice of coordinate system



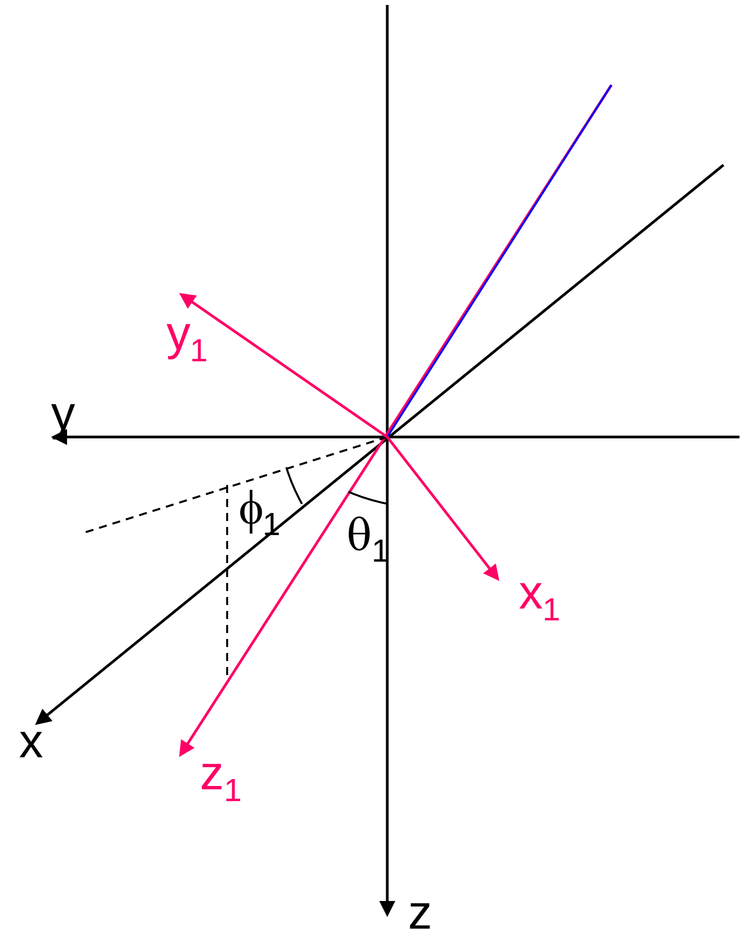
Snell's Law: $$(n_1\hat{r}_1 - n_2\hat{r}_2 + \nabla\Phi)\times\hat{n} = 0$$

Coordinate system

- The z-axis of the refractive/diffractive surface is along the normal of the surface pointing to medium 2. The x and y axes are arbitrarily chosen.

- The z-axes of the incident and refracted/diffracted wavefronts are along the principal rays. The principal ray vectors are $\mathbf{z_1}=(\sin\theta_1\cos\phi_1, \sin\theta_1\sin\phi_1, \cos\theta_1)$ and $\mathbf{z_2}=(\sin\theta_2\cos\phi_2, \sin\theta_2\sin\phi_2, \cos\theta_2)$, respectively, in the surface coordinate frame.

- The x-axis of the incident wavefront is chosen to be perpendicular to the incidence plane, i.e. $\mathbf{x_1} =\mathbf{z_1}\mathrm{x}\mathbf{z}$, then $\mathbf{y_1}$ is determined following the right-hand rule. The x-y axes of the refracted/diffracted wavefront is chosen similarly, i.e. $\mathbf{x_2}=\mathbf{z_2}\mathrm{x}\mathbf{z}$, and $\mathbf{y_2} =\mathbf{z_2}\mathrm{x}\mathbf{x_2}$.



Derivation

- Trace the principal ray with the general Snell's law
- For a given incident wavefront, calculate the adjacent ray's incidence point on the surface and angle
- Differentiate the Snell's law to get the adjacent ray's angle after refraction/reflection/diffraction at the surface
- Get the wavefront info from the ray's position and angle
- Extract the relations between the incoming and outgoing wavefronts in the vicinity of the principal ray

Differentiating Snell's Law:

$$(n_1 d\hat{r}_1 - n_2 d\hat{r}_2 + d\nabla\Phi) \times \hat{n} + (n_1 \hat{r}_1 - n_2 \hat{r}_2 + \nabla\Phi) \times d\hat{n} = 0$$

See details in "*Generalized Coddington Equations*" by C. Zhao and J. H. Burge, to be submitted to Optics Express.

Coddington Equations – further generalized

$$
\begin{aligned}
& n_2 \begin{bmatrix} \sin\phi_2 & \cos\phi_2 \cos\theta_2 \\ -\cos\phi_2 & \sin\phi_2 \cos\theta_2 \end{bmatrix} \cdot \begin{bmatrix} C_{xx2} & C_{xy2} \\ C_{xy2} & C_{yy2} \end{bmatrix} \cdot \begin{bmatrix} \sin\phi_2 & -\cos\phi_2 \\ \cos\phi_2 \cos\theta_2 & \sin\phi_2 \cos\theta_2 \end{bmatrix} - \\
& n_1 \begin{bmatrix} \sin\phi_1 & \cos\phi_1 \cos\theta_1 \\ -\cos\phi_1 & \sin\phi_1 \cos\theta_1 \end{bmatrix} \cdot \begin{bmatrix} C_{xx1} & C_{xy1} \\ C_{xy1} & C_{yy1} \end{bmatrix} \cdot \begin{bmatrix} \sin\phi_1 & -\cos\phi_1 \\ \cos\phi_1 \cos\theta_1 & \sin\phi_1 \cos\theta_1 \end{bmatrix} \\
& = (n_2 \cos\theta_2 - n_1 \cos\theta_1) \begin{bmatrix} C_{xxr} & C_{xyr} \\ C_{xyr} & C_{yyr} \end{bmatrix} + \begin{bmatrix} C_{xxd} & C_{xyd} \\ C_{xyd} & C_{yyd} \end{bmatrix}
\end{aligned}
$$

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Coddington Equations – further generalized

$$\begin{aligned}
&n_2(C_{xx2}\sin^2\phi_2 + C_{xy2}\sin 2\phi_2\cos\theta_2 + C_{yy2}\cos^2\phi_2\cos^2\theta_2)\\
&-n_1(C_{xx1}\sin^2\phi_1 + C_{xy1}\sin 2\phi_1\cos\theta_1 + C_{yy1}\cos^2\phi_1\cos^2\theta_1)\\
&=(n_2\cos\theta_2 - n_1\cos\theta_1)C_{xxr} + C_{xxd}
\end{aligned}$$

$$\begin{aligned}
&n_2(C_{xx2}\cos^2\phi_2 - C_{xy2}\sin 2\phi_2\cos\theta_2 + C_{yy2}\sin^2\phi_2\cos^2\theta_2)\\
&-n_1(C_{xx1}\cos^2\phi_1 - C_{xy1}\sin 2\phi_1\cos\theta_1 + C_{yy1}\sin^2\phi_1\cos^2\theta_1)\\
&=(n_2\cos\theta_2 - n_1\cos\theta_1)C_{yyr} + C_{yyd}
\end{aligned}$$

$$\begin{aligned}
&n_2(-1/2\cdot C_{xx2}\sin 2\phi_2 - C_{xy2}\cos 2\phi_2\cos\theta_2 + 1/2\cdot C_{yy2}\sin 2\phi_2\cos^2\theta_2)\\
&-n_1(-1/2\cdot C_{xx1}\sin^2\phi_1 - C_{xy1}\cos 2\phi_1\cos\theta_1 + 1/2\cdot C_{yy1}\sin 2\phi_1\cos^2\theta_1)\\
&=(n_2\cos\theta_2 - n_1\cos\theta_1)C_{xyr} + C_{xyd}
\end{aligned}$$

Verification

- A Zemax model is created to test the further generalized Coddington Equations
  - A incident wavefront whose curvature matrix can be arbitrarily set
  - A refractive/diffractive surface whose surface and phase curvature matrices can be arbitrarily set
  - The Principal ray incident angle can be arbitrarily set
  - For any combination of the above parameters, use the further generalized Coddington Equations to calculate the outgoing wavefront's curvature matrix which is then translated to two line foci locations and angle
  - In Zemax, set the image plane to the predicted line image locations and see the images. **The line images are right there with the predicted angles!**

Application

- The computer generated hologram (CGH), a diffractive element, is used as null lens in optical testing
- It forms intermediate images of the test surface; these images are then relayed to the camera by the interferometer optics
- Aberrations of the intermediate images, primarily astigmatism and field curvatures, affect measurements of detailed features on the test surface
- Coddington Equations are the tools for analyzing the intermediate image quality

C. Zhao and J. H. Burge, "Imaging aberrations from null correctors", Proc. SPIE, Vol. 6723 67230L

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GMT null test

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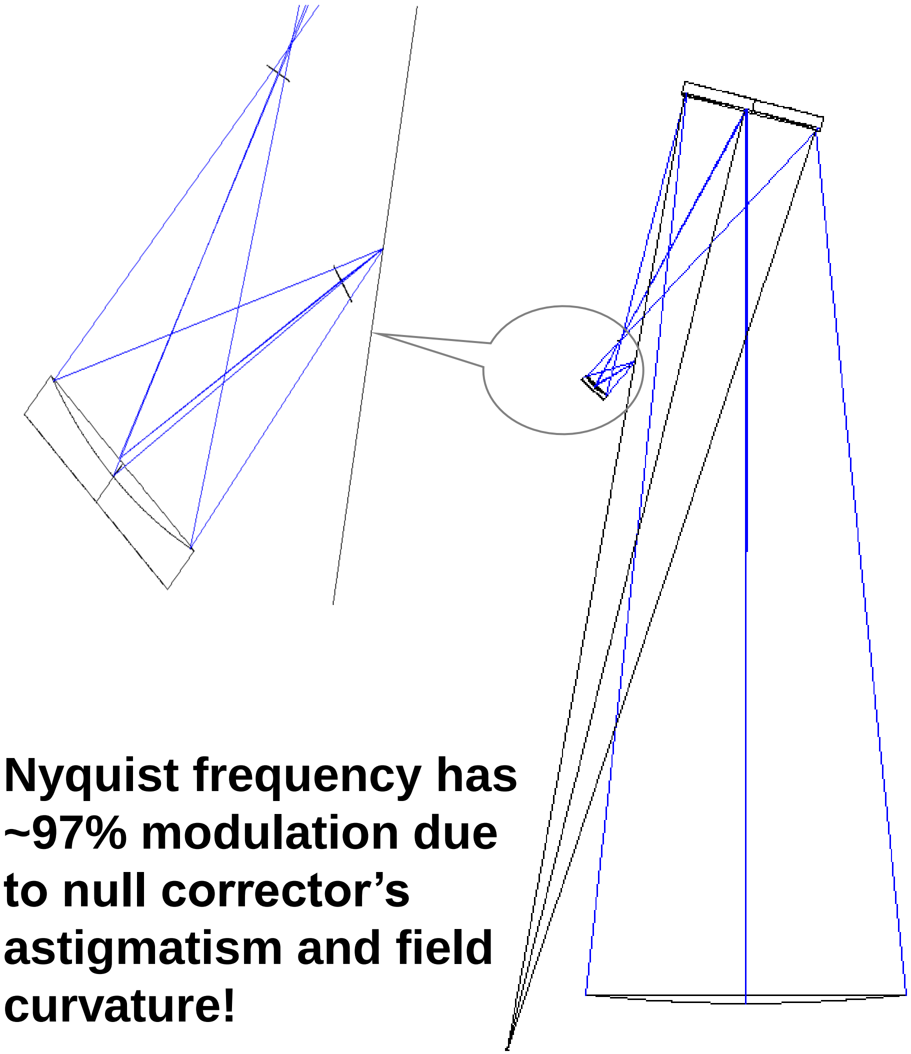
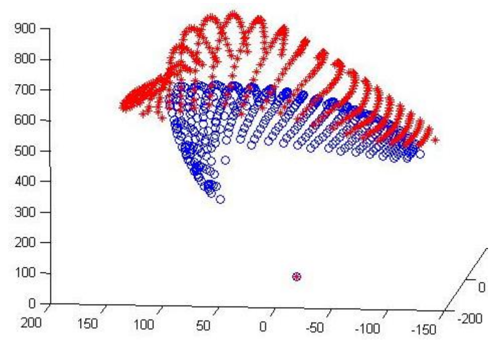
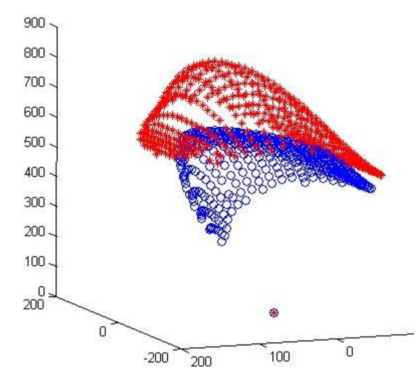


Field surfaces:





Nyquist frequency has ~97% modulation due to null corrector's astigmatism and field curvature!

Summary

- Coddington Equations were further generalized. They apply to all types of image forming surfaces: refractive, reflective and diffractive
- The equations were verified with a general model set up in Zemax
- One application is to calculate the image locations of the test surface through the CGH null in an interferometric test