# Relating axial motion of optical elements to focal shift

Katie Schwertz and J. H. Burge

College of Optical Sciences, University of Arizona, Tucson AZ 85721, USA

katie.schwertz@gmail.com

## ABSTRACT

In this paper, simple relationships are presented to determine the amount of focal shift that will result from the axial motion of a single element or group of elements in a system. These equations can simplify first-order optomechanical analysis of a system. Examples of how these equations are applied are shown for lenses, mirrors, and groups of optical elements. Limitations of these relationships are discussed and the accuracy is shown in relation to modeled systems.

**Keywords:** focal shift, optomechanics, axial motion, defocus, alignment

## 1. INTRODUCTION

When designing and tolerancing an optical system, typically a full computer model of the system is created and a complete tolerance analysis is done. It is good practice to understand what the expected outcome of a simulation is before dedicating time to the endeavor. First order calculations, estimations, rules of thumb, and experience can all provide means with which to determine an expected outcome[1-2].

The equations presented in this paper provide a simple way to estimate the amount of focal shift that will occur from the axial motion of an optical element, or group of elements, in a system based on the cone of light entering and exiting the element(s). Related equations exist for determining the amount of image motion due to the decenter and tilt of an optical element[3]. These equations can be used to estimate the tolerances needed on the axial position of an optical element or the expected amount of focal shift from a given axial motion. They can also be used to determine which element in a system will affect focus the most.

### 1.1 Sign Convention

All calculations and formulas in this paper follow the sign convention as presented in *Field Guide to Geometrical Optics*[4]. All angles are defined relative to a reference line, where counterclockwise angles are positive and clockwise angles are negative. The reference line in this case is the optical axis (by convention, the z-axis).
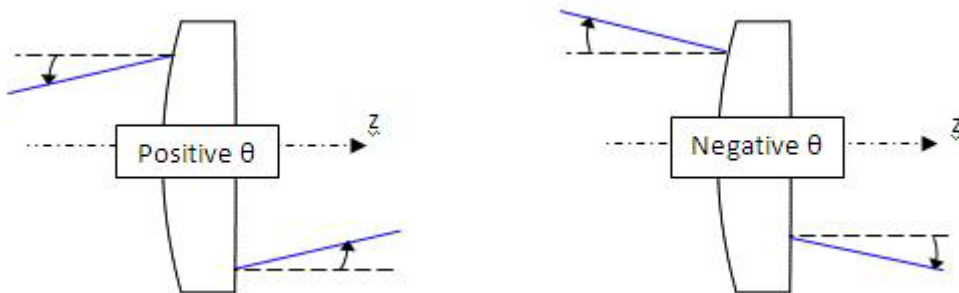


Figure 1: Sign convention for defining angles

All distances are defined relative to a reference point or plane, where distances (arrows) to the right and above the reference point are positive and distances (arrows) to the left and below the reference point are negative.
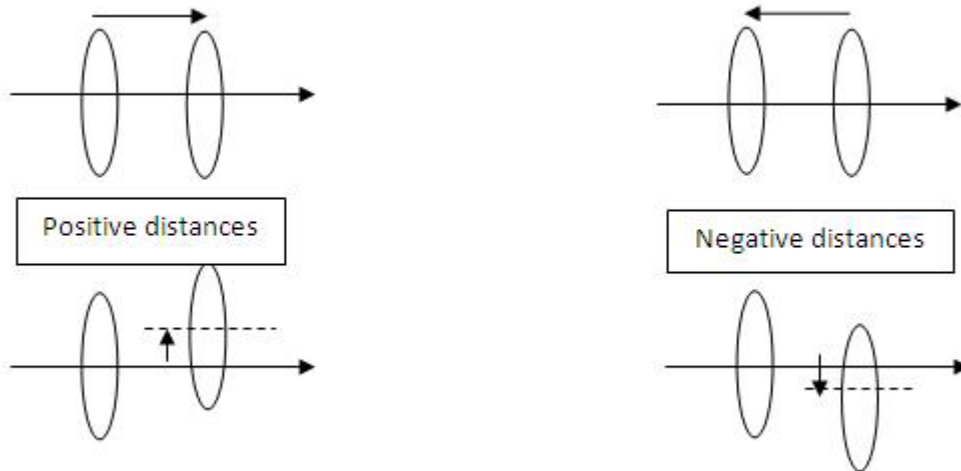
Figure 2: Sign convention for defining distances

## 2. IMAGE MOTION EQUATIONS

### 2.1 Image motion for refractive surfaces in terms of magnification

Consider a biconvex lens at finite conjugates that is shifted in the $+\hat{z}$ direction by an amount, $\Delta z_L$. As a result, the focus will shift a given amount, $\Delta z_f$.
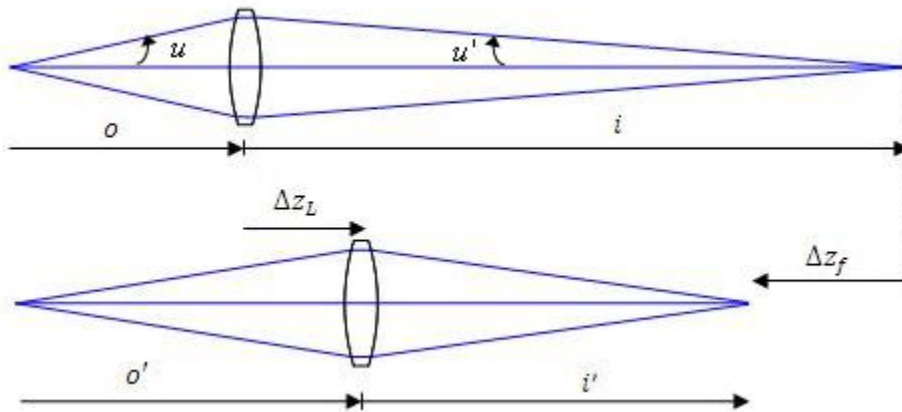


Figure 3: Shifting an optical element by an amount, $\Delta z_L$, will cause the focus to shift by $\Delta z_f$.

Using the geometry of the system above, the following relationships can be defined:

$$o' = o + \Delta z_L \qquad \textbf{Equation 1}$$

$$i' = i - \Delta z_L + \Delta z_f \qquad \textbf{Equation 2}$$

Rearranging these two equations yields:

$$\Delta o = o - o' = -\Delta z_L \qquad \textbf{Equation 3}$$

$$\Delta i = i - i' = \Delta z_L - \Delta z_f \qquad\qquad \textbf{Equation 4}$$

From the imaging equation,

$$\frac{1}{i} + \frac{1}{o} = \frac{1}{f} \qquad\qquad \textbf{Equation 5}$$

where $f$ is the focal length of the lens, we can differentiate to obtain,

$$\frac{-\partial i}{i^2} + \frac{-\partial o}{o^2} = 0 \qquad\qquad \textbf{Equation 6}$$

When $\partial i$ and $\partial o$ are small, we can make the estimation that $\partial i \approx \Delta i$ and $\partial o \approx \Delta o$. Rearranging Equation 6 we find,

$$\frac{\Delta i}{\Delta o} = -\frac{i^2}{o^2} = -m^2 \qquad\qquad \textbf{Equation 7}$$

where $m$ is the magnification of the lens. Inserting Equations 3 and 4 from above and rearranging we find the expression:

$$\Delta z_f = \Delta z_L \left(1 - m^2\right) \qquad\qquad \textbf{Equation 8}$$

This derivation can then be extended to a system with two lenses by knowing the shift in the image of the first lens ($\Delta z_{f1}$) serves as the shift in the object for the second lens ($\Delta z_{L2}$). The resulting relationship is found to be:

$$\Delta z_f = \Delta z_L \left(1 - m_1^2\right) \cdot m_2^2 \qquad\qquad \textbf{Equation 9}$$

where $m_1$ is the magnification of the first lens and $m_2$ is the magnification of the second lens. Again, this expression can be extended to a system with $N$ lenses where element $x$ undergoes an axial shift. The expression is simply multiplied by the magnification of the individual lenses following the shifted element. Lenses before the shifted element will not be affected.

$$\Delta z_f = \Delta z_L \left(1 - m_x^2\right)\left(m_{x+1}^2\right)\left(m_{x+2}^2\right)\ldots\left(m_N^2\right) \qquad\qquad \textbf{Equation 10}$$

## 2.2 Image motion for refractive surfaces in terms of numerical aperture

The magnification of an element can be defined in terms of the paraxial marginal ray angle, $u$, and the index of refraction, $n$, surrounding the lens. The angles are defined as they occur at the element in a system, not at infinite conjugates.
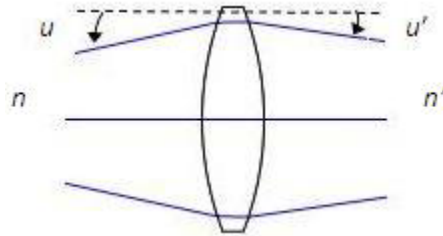


Figure 4: The marginal ray angles at a specific element

$$m = \frac{nu}{n'u'}$$ **Equation 11**

The numerical aperture (NA) defines the cone of light entering or exiting an optic in terms of the real marginal ray angle, $U$, and can be approximated using the marginal ray angle. This approximation is valid for slow systems (small ray angles) and systems that have little or no pupil aberrations.

$$NA = n \sin U \approx nu$$ **Equation 12**

The magnification can then be written in terms of NA:

$$m = \frac{NA}{NA'}$$ **Equation 13**

We can substitute this expression for magnification into Equation 10 to find:

$$\Delta z_f = \Delta z_L \left[ 1 - \left( \frac{NA_x}{NA_{x'}} \right)^2 \right] \left( \frac{NA_{x+1}}{NA_{(x+1)'}} \right)^2 \left( \frac{NA_{x+2}}{NA_{(x+2)'}} \right)^2 \cdots \left( \frac{NA_N}{NA_{im}} \right)^2$$ **Equation 14**

where $NA_{im}$ is the numerical aperture at the image plane. The light exiting each element will enter the following element at the same angle, meaning that $NA_{x'} = NA_{x+1}$, $NA_{(x+1)'} = NA_{x+2}$, etc. The equation then reduces to:

$$\Delta z_f = \Delta z_L \left( \frac{NA_{x'}^2 - NA_x^2}{NA_{im}^2} \right)$$ **Equation 15**

### 2.3 Image motion for refractive surfaces in terms of lens focal length and beam diameter

The angle of light exiting a lens can be expressed in terms of the angle of the incoming light and the lens focal length and beam diameter incident on the optic by the paraxial raytrace equation:

$$n'u' = nu - \frac{y}{f}$$ **Equation 16**

$f$ = focal length of lens
$y$ = height of light incident on the optic (marginal ray height)

Again using the approximation for NA in Equation 12 and inserting this expression into Equation 15, we can derive another equation for focal shift based on the properties of the lens:

$$\Delta z_f = \frac{\Delta z_L}{NA_{im}^2} \left[ \frac{y}{f} \left( \frac{y}{f} - 2NA_x \right) \right]$$ **Equation 17**

However, after comparing the results of the expression above to the actual results from a ray trace model, it was found that this equation has very large error in comparison to the prior two equations presented. It is included here to provide insight as to the sensitivity of a given optic based on the focal length, height of the beam on the optic, and angle of light entering the optic. This equation behaves differently for three different regions – where $y/f \gg 2NA_x$,

$y/f \ll 2NA_x$, or $y/f \approx 2NA_x$. The plots below show the ratio of focal shift to element shift as the ratio of $f/2y$ increases (i.e. increasing f/#) for the different regions. When $y/f \approx 2NA_x$, the ratio is approximately zero.
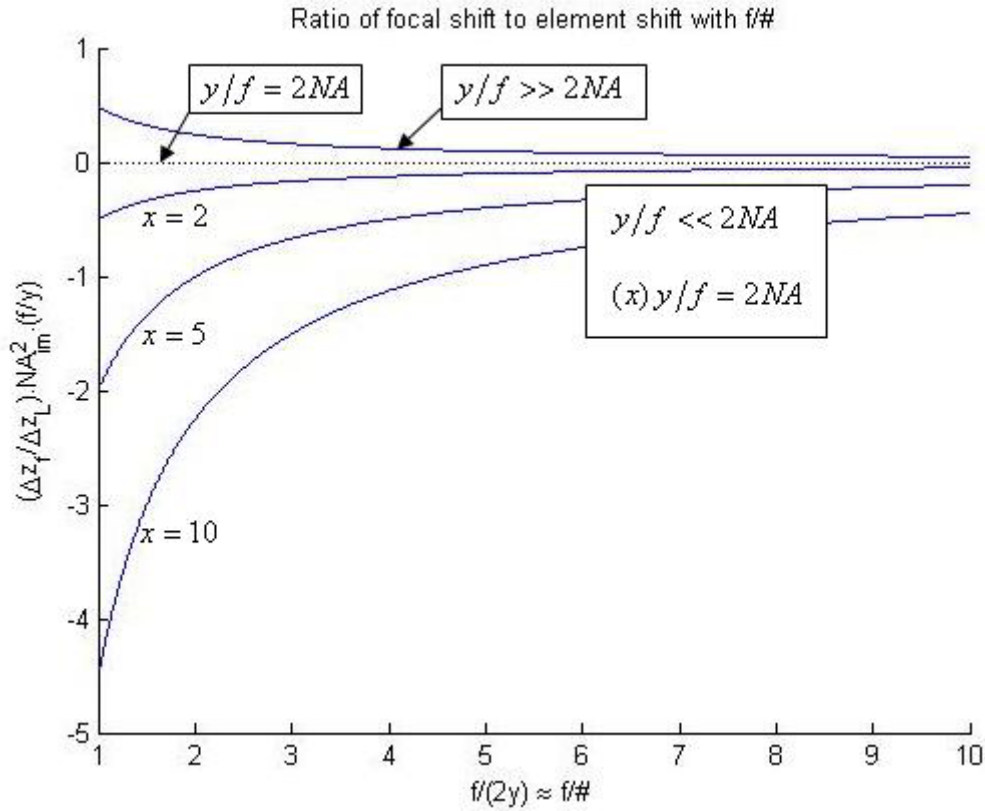


Figure 5: Amount of focal shift due to element shift based on the f/# of the element

These graphs can be used to get a general idea of the amount of focal shift that will occur for a given element motion based on the element focal length, beam diameter, and entering NA. The most sensitive elements of a system can then easily be determined. As evident by the graph, elements with a large f/# will be very insensitive to element axial motion.

### 2.4 Image motion for reflective surfaces
The equations derived above can also be applied to reflective surfaces, but with a sign correction. Due to the fact that the numerical aperture terms are squared, the change in direction of the light after reflection (and therefore the change in sign) does not affect the equation. For reflective surfaces, the sign in Equation 10 and Equation 15 must be changed from subtraction to addition:

$$\Delta z_f = \Delta z_M \left(1 + m_x^2\right)\left(m_{x+1}^2\right)\left(m_{x+2}^2\right)...\left(m_N^2\right) \qquad \textbf{Equation 18}$$

$$\Delta z_f = \Delta z_M \left( \frac{NA_{x'}^2 + NA_x^2}{NA_{im}^2} \right) \qquad \textbf{Equation 19}$$

where $\Delta z_M$ is the axial motion of the mirror.

# 3. ERROR ANALYSIS

For small focal lengths and large incident beam diameters (i.e. fast $f/\#$ lenses), the relationships presented here have greater error. As seen from Equation 15, and knowing that

$$f/\# \approx \frac{1}{2NA_{im}} \; , \qquad\qquad \textbf{Equation 20}$$

the error in these equations will be proportional to the $(f/\#)^2$ of the lens. The wavefront error due to defocus can also be found. Defocus is given by the expression[4]:

$$\Delta W_{20} = \frac{\Delta z_f}{8(f/\#)^2} \qquad\qquad \textbf{Equation 21}$$

Substituting in Equations 15 and 20, we find:

$$\Delta W_{20} = \frac{\Delta z_L}{2}\left(NA_{x'}^2 - NA_x^2\right) \qquad\qquad \textbf{Equation 22}$$

This allows for direct calculation of the amount of defocus in the image based on the axial motion and the entering and exiting NA values for the given element.

# 4. BASIC EXAMPLES

A series of examples are shown below for simple systems in which the relationship between the optical element motion and the focal shift is already known, or can be easily determined. For each example shown, the element motion is in the positive direction, so the sign of $\Delta z_L$ is taken as positive. If the optical element motion is in the $-\hat{z}$ direction, the sign of $\Delta z_L$ is taken as negative.

## 4.1 Positive Lens with Object at Infinity

For a refractive surface with an object at infinity, the light will focus at the focal length of the lens and the magnification is zero. If we move the lens axially an amount $\pm\Delta z$, the image will move the same distance, in the same direction.
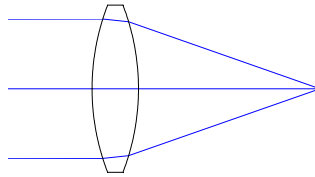


Figure 6: Positive lens with object at infinity (m = 0)

Using Equation 8 we find:

$$\Delta z_f = \Delta z_L\left(1 - m^2\right) = \Delta z_L(1 - 0^2) = \Delta z_L$$

Using Equation 15 and knowing that light is entering the lens parallel to the optical axis $\left(NA_x = 0\right)$ and the NA in image space is the same as $NA_{x'}$, we find:

$$\Delta z_f = \Delta z_L \left( \frac{NA_{x'}^2 - NA_x^2}{NA_{im}^2} \right) = \left( \frac{NA_{x'}^2 - 0}{NA_{x'}^2} \right) = \Delta z_L$$

## 4.2 Powered Mirror with Object at Infinity

Similar to the lens above, the magnification of a powered mirror with an object at infinity is zero. Since light is entering parallel to the optical axis, $NA_x$ is zero and the NA in image space is the same as $NA_{x'}$.
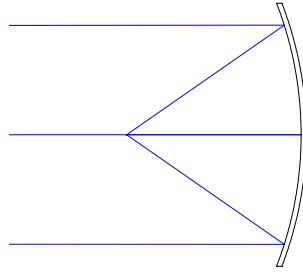


Figure 7: Concave mirror with object at infinity (m = 0)

Using Equation 18 we find:

$$\Delta z_f = \Delta z_M \left( 1 + m^2 \right) = \Delta z_M (1 + 0^2) = \Delta z_M$$

Using Equation 19 we find:

$$\Delta z_f = \Delta z_M \left( \frac{NA_{x'}^2 + NA_x^2}{NA_{im}^2} \right) = \left( \frac{NA_{x'}^2 + 0}{NA_{x'}^2} \right) = \Delta z_M$$

## 4.3 Plane Parallel Plate

Inserting a plane parallel plate into a beam will displace the beam axially, but once inserted, moving it axially will not cause additional image motion. The angle of light exiting the plate is the same as the angle of light entering it, so the magnification of the plate is 1.
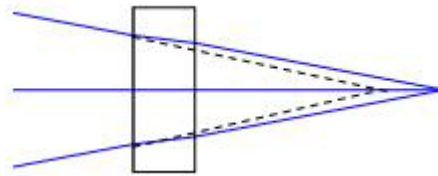


Figure 8: Plane parallel plate in a converging beam (m = 1)

Using Equation 8 we find:

$$\Delta z_f = \Delta z_L \left( 1 - m^2 \right) = \Delta z_L (1 - 1^2) = 0$$

Using Equation 15 we find:

$$\Delta z_f = \Delta z_L \left( \frac{NA_{x'}^2 - NA_x^2}{NA_{im}^2} \right) = \left( \frac{NA_x^2 - NA_x^2}{NA_{im}^2} \right) = 0$$

### 4.4 Flat Mirror

Inserting a flat mirror into a beam will fold the optical path, but will not add power. The magnification of a flat mirror is 1 and the focus will shift twice the amount that the mirror is shifted axially. From the law of reflection we know the angle of the light before and after reflection will be the same $(NA_{x'} = NA_x)$.
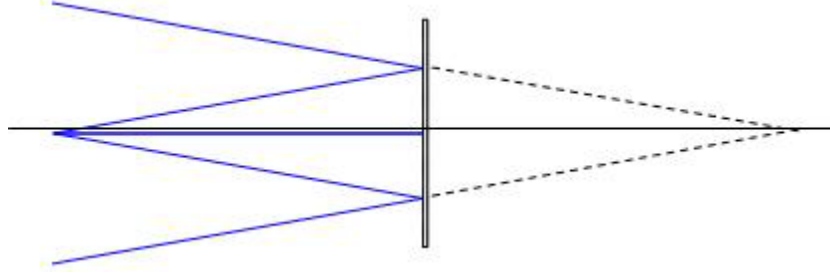


Figure 9: Flat mirror in a converging beam (m = 1)

Using Equation 18 we find:

$$\Delta z_f = \Delta z_M \left(1 + m^2\right) = \Delta z_M \left(1 + 1^2\right) = 2\Delta z_M$$

Using Equation 19 we find:

$$\Delta z_f = \Delta z_M \left(\frac{NA_{x'}^2 + NA_x^2}{NA_{im}^2}\right) = \left(\frac{2NA_x^2}{NA_x^2}\right) = 2\Delta z_M$$

## 5. VERIFICATION OF EQUATIONS

### 5.1 Verification of individual lens motion in a Cooke triplet

To verify the validity of the relationships presented above, individual elements were perturbed in a Cooke triplet. The sample Cooke triplet design provided with the Zemax software was used[5]. The distance to the object was set to 200mm (instead of infinity) to demonstrate the validity of the equations for a finite conjugate system. A summary of the relevant system properties is given below.

Table 1: Cooke Triplet System Data

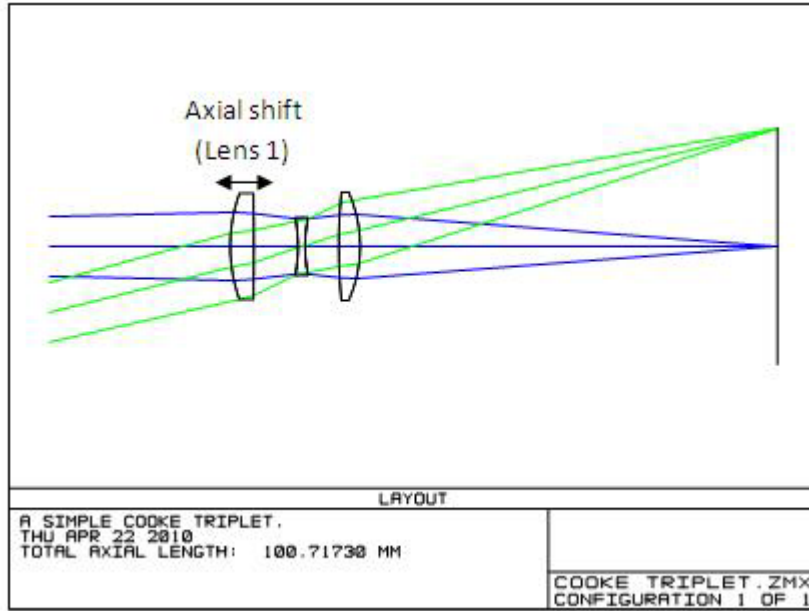| Parameter | Value |
|---|---|
| Effective Focal Length | 50.14 mm |
| Working F/# | 6.5 |
| Entrance Pupil Diameter | 10 mm |
| Image Space NA | 0.0762 |
| Lens 1 focal length | 33.75 mm |
| Lens 2 focal length | -16.85 mm |
| Lens 3 focal length | 24.28 mm |

Figure 10: Cooke Triplet design used to compare the actual focal shift to the predicted values for individual lenses

Each individual lens was shifted axially in the $+\hat{z}$ direction while the other two lenses were held in a fixed position. The shift of the marginal ray focus was recorded for each lens shift. The graph below compares the predicted focal shift from Equation 15 to the actual focal shift recorded from the Zemax simulation.
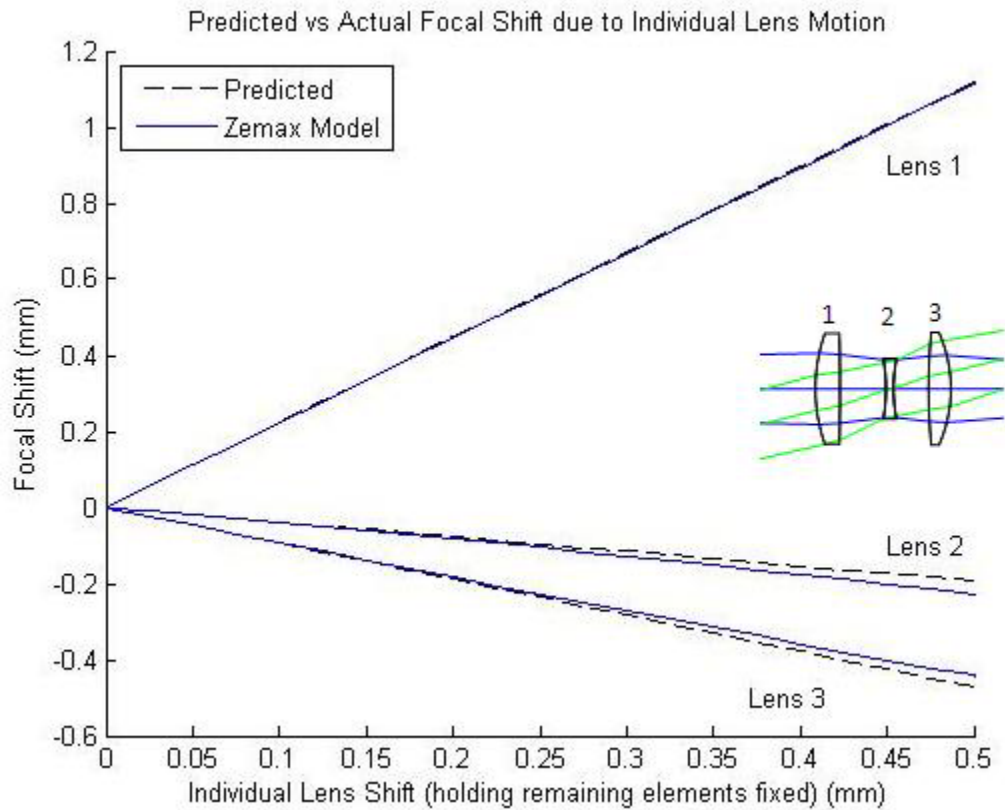


Figure 11: Graph comparing predicted vs actual focal shift for an individual lens

Because these equations are derived from geometrical optics principals, they are valid for small shifts and are more accurate for lower powered surfaces (i.e. larger incidence angles make the paraxial assumptions less valid). This can be seen in the graph, where Lens 1 has less power (longer focal length) than Lens 2 or Lens 3.

## 5.2 Verification of group lens motion in a Double Gauss System

The relationships above can also be applied to a group of elements in a system that shift together. The sample Double Gauss system provided with the Zemax software was used to demonstrate the validity of the equations for the shift of a group of elements. A summary of the relevant system properties is given below.

Table 2: Double Gauss System Data

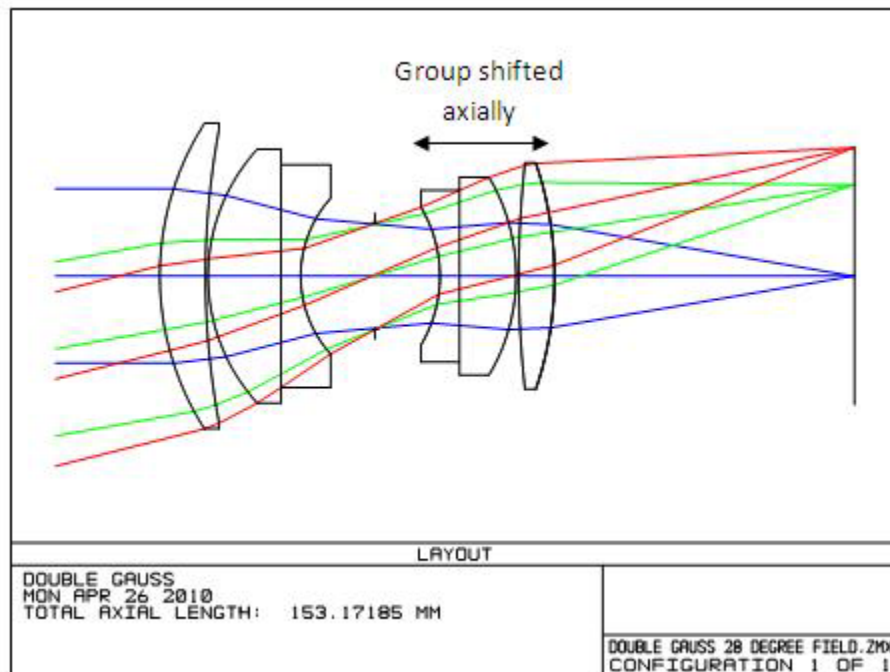| Parameter | Value |
|-----------|-------|
| Effective Focal Length | 99.5 mm |
| Working F/# | 3 |
| Entrance Pupil Diameter | 33.3 mm |
| Image Space NA | 0.165 |
| Focal length of shifted group | 178.6 mm |



Figure 12: Double Gauss design used to compare the actual focal shift to the predicted values for a group of elements

The angle of the light entering the first optic and the angle of the light exiting the final optic are used for the calculations.
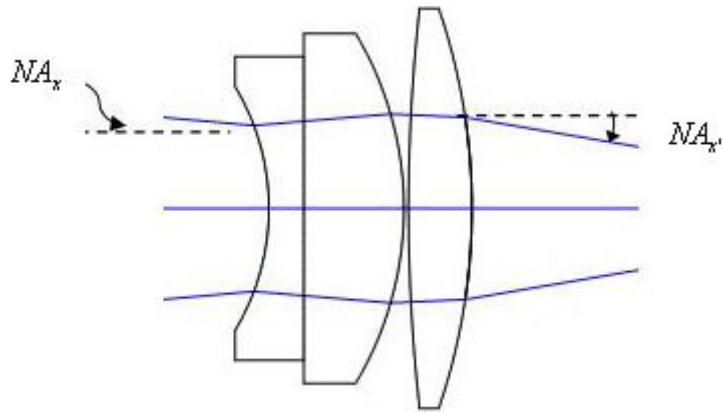
Figure 13: Numerical apertures used for the following calculations

The group of lenses was shifted axially in the $+\hat{z}$ direction and the shift of the marginal ray focus was recorded. The predicted focal shift from Equation 15 was then compared to the shift of the marginal ray focus in the ray trace simulation.
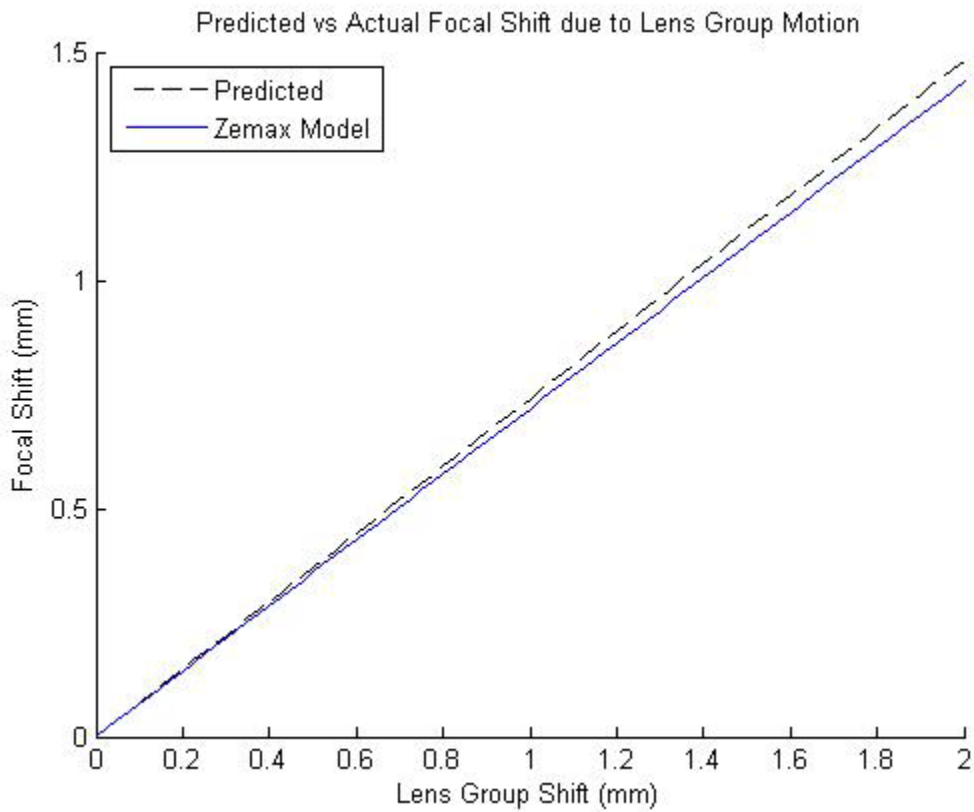


Figure 14: Graph comparing predicted vs actual focal shift for a group of lenses

Since the effective focal length of the shifted group of elements is much larger than each of the individual elements in the Cooke triplet, the predicted values are accurate for larger shifts (>2mm compared to >0.5mm for the Cooke triplet)

**5.3 Verification of individual mirror motion in a Ritchey-Chretien telescope**

To verify the validity of the relationships given for reflective elements, the primary and secondary mirrors in a Ritchey-Chretien telescope were individually shifted axially while the other element remained fixed. The sample Cassegrain-type Ritchey-Chretien telescope provided with the Zemax software was used. A summary of the relevant system properties is given below.

Table 3: Ritchey-Chretien system data

| Parameter | Value |
|---|---|
| Effective Focal Length | 1752.1 mm |
| Working F/# | 11.7 |
| Entrance Pupil Diameter | 150 mm |
| Image Space NA | 0.043 |
| Primary mirror focal length | 371.4 mm |
| Secondary mirror focal length | -145.1 mm |



LAYOUT

CASSEGRAIN RITCHEY-CHRETIEN
TUE APR 27 2010
TOTAL AXIAL LENGTH: 489.20814 MM

CASSEGRAIN-TYPE RITCHIE CRETIEN_PRIMARY.ZMX
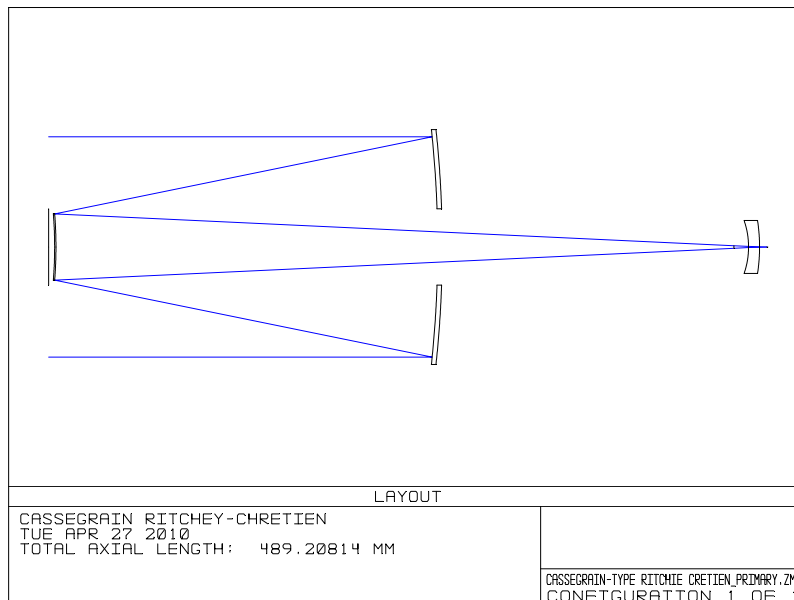CONFIGURATION 1 OF 1

Figure 15: Ritchey-Chretien design used to compare the actual focal shift to the predicted values for individual mirrors

The graph below shows the focal shift predicted by Equation 15 compared to the actual focal shift from Zemax.
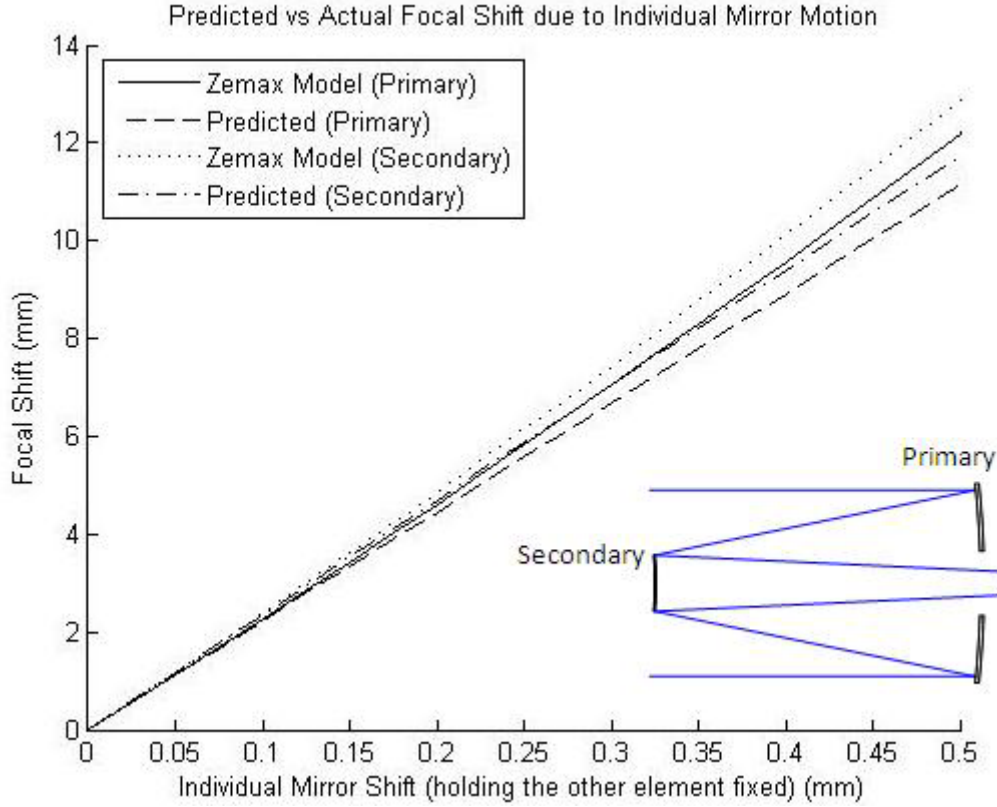
Figure 16: Graph comparing predicted vs actual focal shift for a mirror

As discussed previously, the amount of error in the estimation is proportional to the $(f/\#)^2$ of the shifted elements. The $f/\#$ of the mirrors are larger than the previous two examples, so the error in the estimation is greater here for the same amount of axial motion.

## 6. CONCLUSION

Presented in this paper are simple equations for estimating the focal shift from the motion of an optical element or group of elements. In summary:

$$\Delta z_f = \Delta z_E \left( \frac{NA_{x'}^2 \pm NA_x^2}{NA_{im}^2} \right)$$

$$\Delta z_f = \Delta z_E \left( 1 \pm m_x^2 \right)\left( m_{x+1}^2 \right)\left( m_{x+2}^2 \right)\ldots\left( m_N^2 \right)$$

− for refractive surfaces
+ for reflective surfaces

$\Delta z_f$ = focal shift from nominal

$\Delta z_E$ = axial shift of the element(s)

$NA_x$ = entering numerical aperture (estimated by angle of light entering element, relative to the optical axis)

$NA_{x'}$ = exiting numerical aperture (estimated by angle of light exiting element, relative to the optical axis)

$NA_{im}$ = numerical aperture in the image plane

$m$ = magnification of the element

These equations are valid for small motions and the amount of error from actual focal shift decreases proportional to $1/(f/\#)^2$ of an element. Determining the amount of wavefront error due to defocus was also presented. These equations can be applied to determine which elements in a system affect focal shift the greatest amount or for simplified first order tolerance analysis of how the axial motion of a lens, mirror, or group of lenses is coupled to focal shift.

## REFERENCES

[1] Friedman, Edward and John Lester Miller., [Photonics Rules of Thumb: Optics, Electro-optics, Fiber Optics, and Lasers], McGraw-Hill, New York, (2004).

[2] Schwertz, Katie, [Useful Estimations and Rules of Thumb for Optomechanics]. Unpublished Master's Thesis, retrieved from http://www.optics.arizona.edu/loft/cos/publication.htm (2010).

[3] Burge, J. H., "An easy way to relate optical element motion to system pointing stability," Proc. SPIE 6288 (2006).

[4] Greivenkamp, J. E., [Field Guide to Geometrical Optics], SPIE, Bellingham, (2004).

[5] Zemax Development Corporation, Bellevue WA. www.zemax.com.