Criteria for correction of all aberrations with quadratic field dependence

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Abstract

Aberrations of imaging systems can be described using a polynomial expansion of the dependence on field position. Aberrations on axis and those with linear field dependence can be calculated and controlled using Fermat's principle and the Abbe Sine Condition. We now present a powerful new set of relationships that fully describe the aberrations that depend on the second power of the field. A simple set of equations, derived using Hamilton's characteristic functions, which we call the Pupil Astigmatism Criteria, use on-axis behavior to evaluate and control all aberrations with quadratic field dependence and arbitrary dependence on the pupil. These relations are explained, validated, and applied to design optical systems that are free of all quadratic field dependent aberrations.

OCIS Codes: 220.1010 Aberration theory, 220.4830 Optical systems design

1. Introduction

The aberration of a general optical system depends on both the field and aperture. For an axially symmetric system, let the field be h and the aperture be ρ. The aberrations can be expanded in the following form[1, 2]:

$$W(\vec{h},\vec{\rho}) = \sum W_{2m+k,2n+k,k}(h^2)^m \cdot (\rho^2)^n \cdot (\vec{h}\cdot\vec{\rho})^k \ , \tag{1}$$

where m, n and k are integers. If aberrations are classified in terms of their field dependence, then we have:

- Spherical aberrations with no field dependence such as W_{040}, W_{060}, W_{080}, etc.
- Linear field-dependent aberrations such as W_{131}, W_{151}, W_{171}, etc.
- Quadratic field-dependent aberrations such as W_{222}, W_{242}, W_{262}, … and W_{220}, W_{240}, W_{260}, etc.

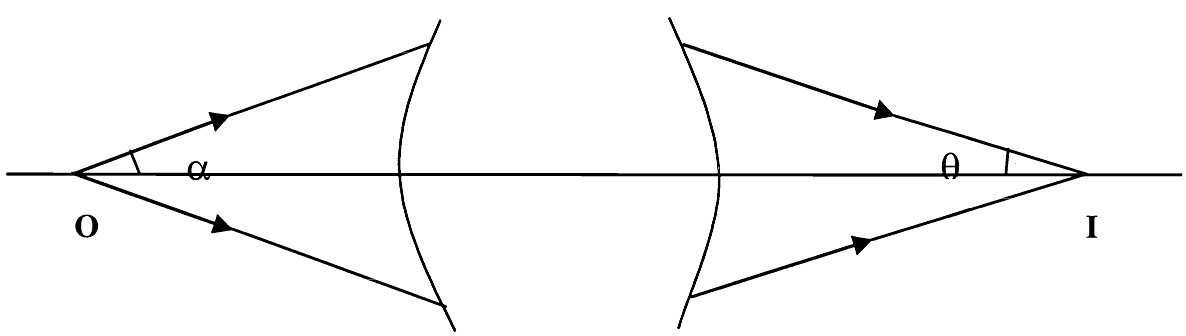

Figure 1. An axisymmetric optical system images the point O to point I

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Criteria already exist for complete correction of aberrations that depend on the field to the 0^{th} (spherical) or 1^{st} order (coma). In Figure 1, the optical system images an on-axis object point *O* to point *I*. For the system to be free of all orders of spherical aberration, Fermat's principle must be satisfied, *i. e.* the optical path length (OPL) from *O* to *I* along any ray must be constant for all α.

$$\text{OPL[OI]} = \text{constant (independent of } \theta), \qquad (2)$$

For the system to be free of all orders of coma, which depends on the field size linearly, the Abbe Sine Condition[1] must be satisfied, i.e.

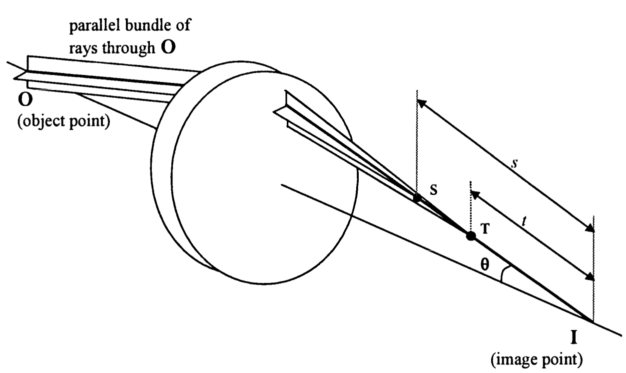
$$\frac{\sin(\alpha)}{\sin(\theta)} = \text{constant (independent of } \theta). \qquad (3)$$

These two conditions are useful in optical design because they tell the designer how to correct all orders of spherical aberrations and coma using only on-axis properties. Likewise, it is desirable to have simple mathematical conditions for correction of other fundamental aberrations. We derived such conditions for correcting the aberrations that are quadratic in field. Like the Sine Condition, these general and elegant conditions involve only the on-axis ray properties.

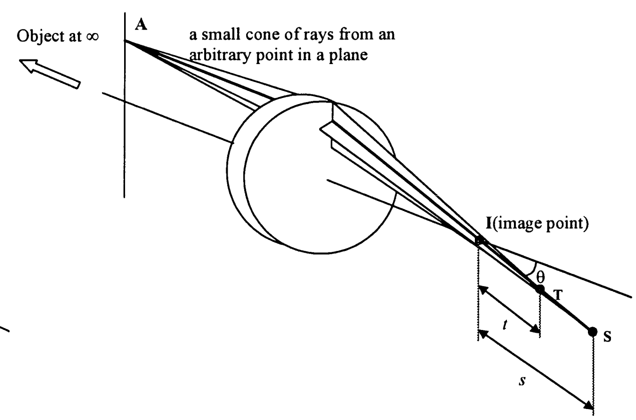
2. The Pupil Astigmatism Criteria

The conditions for correction of aberrations with quadratic field dependence were derived using Hamilton's characteristic functions[2], given in the next section. The Hamiltonian treatment uses ray geometry to specify the *point*, *angle*, and *mixed* characteristic functions. These three functions, which give complete information about the rays and wavefronts are related by a set of differential equations. We developed general forms of these functions for imaging systems and performed a Taylor expansion in terms of the field. We show how the first term, with no field dependence is equivalent to Fermat's principle and the term with first order field dependence can be reduced to the Abbe sine condition. We also show a term that depends on the second power of field, which has profound implications for optical design.

To use this second order term, we evaluate the astigmatism in an infinitesimal bundle of rays about by the object (or image) point, as shown in Fig. 2. For the case of finite conjugates, this bundle originates at a fictitious entrance pupil at infinity and passes the object point collimated. (Since this analysis only requires on-axis ray tracing, the actual pupil position is not important.) In general, we can find the tangential and sagittal focus of this bundle as a function of position in the pupil θ. We define s as the distance from the image point to the sagittal focus and t as the distance to the tangential focus. Similar definitions for s and t are made for the case of an object at infinity by defining the fictitious entrance pupil at any arbitrary plane.



a) Object located a finite distance from optical system



b) Object at infinity

Figure 2. Definition of s and t for imaging systems. The parameter θ gives the pupil dependence.

Using the quadratic term from the expansion of the Hamiltonian relations, we show an elegant set of relations that can be used to control quadratic field dependent aberrations. These are summarized in Table 1.

Table 1. Pupil astigmatism relations for axisymmetrical optical systems.

Type of quadratic field aberrations	Criterion for correction	
Correction of all aberrations in tangential plane	$\dfrac{t(\theta)}{\cos^2(\theta)} = Constant$	(4)
Correction of all aberrations in sagittal plane	$s(\theta) = Constant$	(5)
Set all sagittal aberrations equal to tangential	$s(\theta) = \dfrac{t(\theta)}{\cos^2(\theta)}$	(6)
Correction of all aberrations	$s(\theta) = \dfrac{t(\theta)}{\cos^2(\theta)} = Constant$	(7)

It is important to realize that these criteria give the conditions for correction of quadratic field dependent aberrations ***for all orders of the pupil.*** So, not only are low order astigmatism and field correction corrected, but higher order aberrations of the pupil such as oblique spherical aberration are corrected as well. Also, it is important to understand the distinction between the pupil astigmatism relations and the well known Coddington equations.[3] The Coddington equations can be used for any field angle to determine the tangential and sagittal images defined by an infinitesimal bundle of rays that go through the object point and the very center of the pupil. Thus they allow a complete trace of the aberrations with quadratic pupil dependence and high order field dependence. In contrast, the pupil astigmatism relations give only information about aberrations with quadratic field dependence, but for each point in the pupil. So the Coddington equations are most useful for low numerical aperture, wide field of view systems and the pupil astigmatism conditions are useful for fast systems with a smaller field of view. It should be noticed that t and s that appear in the pupil astigmatism criteria can be calculated by using the Coddington equations, but without the understanding of the criteria, they are not physically significant.

3. Derivation of the Pupil Astigmatism Criteria for systems with object at finite distance

In this section, we reproduce the derivation of the pupil astigmatism relations that is given by Zhao and Burge elsewhere.[4]

a. Hamilton's Characteristic Functions

We used Hamilton's characteristic functions[2, 3] to derive the Pupil Astigmatism Criteria. Hamilton's characteristic functions are a set of functions that represent the optical path length along a ray. In Figure 2a, a ray originates from a point $P_0(x_0, y_0, z_0)$ in object space, and passes through $P_1(x_1, y_1, z_1)$ in image space. O and I are the origins of local coordinate systems. (p_0,q_0,m_0) and (p_1,q_1,m_1) are ray vectors in object space and image space respectively. A ray vector is the vector along the ray with length equal to the index of refraction of the local medium. Q_0 and Q_1 are the intersections of perpendiculars drawn from O and I to the ray in object space and image space respectively. H_0 and H_1 are the intersections of the ray with the x-y plane in object and image space respectively. If we make the following definition:

$[AB]$ = optical path length along the ray from Point A to Point B,

then the Hamilton's characteristic functions are defined as follows:

Point characteristic: $V(x_0,y_0,z_0; x_1,y_1,z_1) = [P_0P_1]$,

Mixed characteristic: $W(x_0,y_0,z_0; p_1,q_1) = [P_0Q_1]$,

Angle characteristic: $T(p_0,q_0; p_1,q_1) = [Q_0Q_1]$.

Hamilton's characteristic functions are very powerful tools for investigation of the general properties of optical systems. If one of the Hamilton characteristics is known, we can obtain all the information about any ray. For example,
if V is known, then

$$p_0 = -\frac{\partial V}{\partial x_0}, \tag{8a}$$

$$p_1 = \frac{\partial V}{\partial x_1}, \tag{8b}$$

if W is known, then

$$p_0 = -\frac{\partial W}{\partial x_0}, \tag{9a}$$

$$X_1 = -\frac{\partial W}{\partial p_1}, \tag{9b}$$

and if T is known, then

$$X_0 = \frac{\partial T}{\partial p_0}, \tag{10a}$$

$$X_1 = -\frac{\partial T}{\partial p_1}. \tag{10b}$$

Other coordinates and ray vector components can be calculated in the same way.

Since the Hamilton's mixed and angle characteristic functions can be used to calculate the ray intercept at a plane, we can then use them to calculate ray aberrations.

b. Derivation of the Pupil Astigmatism Criteria

We use Hamilton's mixed characteristic function W to derive the criteria for the finite conjugate systems. For a rotationally symmetric system, W only depends on the following 3 quantities:

$$\begin{aligned} h^2 &= x_0^{\ 2} + y_0^{\ 2}, \\ \rho^2 &= p_1^{\ 2} + q_1^{\ 2}, \\ \vec{h} \cdot \vec{\rho} &= x_0 p_1 + y_0 q_1. \end{aligned} \tag{11}$$

If we expand W in a power series of the field and neglect 3rd and higher powers, we find

$$W(x_0, y_0, p_1, q_1; z_0, z_1) = W_0(\rho) + (x_0 p_1 + y_0 q_1) W_1(\rho) + (x_0 p_1 + y_0 q_1)^2 W_2(\rho) + (x_0^{\ 2} + y_0^{\ 2}) W_3(\rho), \tag{12}$$

where W_1, W_2 and W_3 are coefficients of the expansion and they are functions of ρ only.
If we consider a field point in the x-z plane, then $y_0 = 0$ and

$$W(x_0, y_0, p_1, q_1; z_0, z_1) = W_0(\rho) + x_0 p_1 W_1(\rho) + x_0^{\ 2}(p_1^{\ 2} W_2(\rho) + W_3(\rho)). \tag{13}$$

According to Eq. (9), the coordinates of the ray intersection with the x-y plane in image space are:

$$x_1 = -\frac{\partial W_0(\rho)}{\partial p_1} - x_0 \frac{\partial}{\partial p_1}(p_1 W_1(\rho)) - x_0^{\ 2} \frac{\partial}{\partial p_1}(p_1^{\ 2} W_2(\rho) + W_3(\rho)) \tag{14}$$

and

$$y_1 = -\frac{\partial W_0(\rho)}{\partial q_1} - x_0 \frac{\partial}{\partial q_1}(p_1 W_1(\rho)) - x_0^{\ 2} \frac{\partial}{\partial q_1}(p_1^{\ 2} W_2(\rho) + W_3(\rho)). \tag{15}$$

Also, according to Eq. (9),

$$p_0 = -p_1 W_1(\rho) \tag{16}$$

at Point O.

We know $\frac{p_0}{p_1}$ is the magnification and generally it is a function of ρ. Let $M(\rho)$ be the magnification, then

$$M(\rho) = -W_1(\rho). \tag{17}$$

The ideal image of a point at $(x_0, 0, z_0)$ will be at $(x_1', 0, z_1)$ where

$$x_1' = M(0) x_0. \tag{18}$$

Let Δx and Δy be the lateral aberrations, and substitute Eq. (18) into Eqs. (14) and (15). We then obtain the following equations for Δx and Δy:

$$\Delta x = x_1 - x_1'$$
$$= -\frac{\partial W_0(\rho)}{\partial p_1} + x_0 \frac{\partial}{\partial p_1}(p_1(M(\rho) - M(0))) - x_0^{\ 2} \frac{\partial}{\partial p_1}(p_1^{\ 2} W_2(\rho) + W_3(\rho)), \tag{19.1}$$

and

$$\Delta y = y_1$$

$$= -\frac{\partial W_0(\rho)}{\partial q_1} + x_0 \frac{\partial}{\partial q_1}(p_1 M(\rho)) - x_0^2 \frac{\partial}{\partial q_1}(p_1^2 W_2(\rho) + W_3(\rho)). \tag{19.2}$$

The first term of the right hand side of equations in (19) gives rise to field-independent spherical aberrations of the form W_{0n0} where n is an even number. The second term gives rise to all the linear field-dependent aberrations of the form W_{1n1} where n is an odd number. The third term gives rise to all the quadratic field-dependent aberrations of the form W_{2n2} and W_{2n0}, where n is an even number. Apparently $W_2(\rho)$ determines W_{2n2}, and $W_3(\rho)$ determines W_{2n0}.

For all the linear field-dependent aberrations to be corrected, $M(\rho)$ must be constant, i.e.

$$M(\rho) = \frac{p_0}{p_1} = \textit{constant}, \tag{20}$$

which is the famous Abbe Sine Condition.

For all the quadratic field-dependent aberrations to be corrected, the following equation must be true:

$$p_1^2 W_2(\rho) + W_3(\rho) = \textit{constant} \;, \tag{21}$$

which implies $W_2(\rho) = 0$ and $W_3(\rho)$ = constant.

We would like to know what $p_1^2 W_2(\rho) + W_3(\rho)$ represents physically. Consider a field point A in the optical system shown in Figure 4(a). A is on the x-axis and has an infinitesimal field height x_0. Now trace a ray from Point O to Point I, which has an angle θ with the optical axis in image space. Then trace a parallel ray from A in the tangential plane. The two rays intersect with each other at T in image space. Draw a perpendicular from I to the ray originating from A, the foot is denoted as B. Let $TI = t$ (>0), then the optical path length from T to I is

$$[TI] = n_i t, \tag{22}$$

where n_i is the index of refraction of medium in image space. The mixed Hamilton's characteristic function for the ray that originates from O is then

$$W(O) = [OT] + [TI], \tag{23}$$

and the mixed Hamilton's characteristic function for the ray that originates from A is then

$$W(A) = [AT] + [TB], \tag{24}$$

where

$$[AT] = [OT] - x_0 p_0, \tag{25}$$

and

$$[TB] = [TI]\cos(\delta\theta)$$

$$\cong [TI] - [TI](\delta\theta)^2 / 2\,. \tag{26}$$

Combining Eqs. (23), (24), (25) and (26), we get

$$W(A) = W(O) - x_0 p_0 - [TI](\delta\theta)^2 / 2\,, \tag{27}$$

where

$$\delta\theta = x_1 \cos(\theta) / t = x_0 M(\rho)\cos(\theta) / t\,, \tag{28}$$

where $M(\rho)$ is the magnification.

Then

$$W(A) = W(O) - x_0 p_0 - x_0^{\,2}(n_i M^2(\rho)\cos^2(\theta)/(2t))\,. \tag{29}$$

Comparing this equation to Eq. (13), we get

$$p_1^{\,2} W_2(\rho) + W_3(\rho) = -\frac{n_i M^2(\rho)\cos^2(\theta)}{2t} \tag{30}$$

in the tangential plane.

To find out what $W_3(\rho)$ is, we trace a ray from Point O to Point I in sagittal plane, which has an angle θ with the optical axis in the image space (see Figure 4(b)). Then trace a parallel ray from A to the image space. The two rays intersect with each other at S in image space. Draw a perpendicular from I to the ray originating from A, and denote the foot as B. Let SI = s (>0). Then the optical path length from S to I is

$$[SI] = n_i s\,. \tag{31}$$

The mixed Hamilton's characteristic function for the ray that originates from O is

$$W(O) = [OS] + [SI], \tag{32}$$

and the mixed Hamilton's characteristic function for the ray that originates from A is

$$W(A) = [AS] + [SB], \tag{33}$$

where

$$[AS] = [OS], \tag{34}$$

and

$$[SB] = [SI]\cos(\delta\theta)$$

$$\cong [SI] - [SI](\delta\theta)^2/2\,. \tag{35}$$

So combining Eqs. (32), (33), (34) and (35), we get

$$W(A) = W(O) - [SI](\delta\theta)^2/2\,, \tag{36}$$

where

$$\delta\theta = \frac{x_1}{s} = \frac{x_0 M(\rho)}{s}\,. \tag{37}$$

Then

$$W(A) = W(O) - x_0{}^2\left(\frac{n_i M^2(\rho)}{2s}\right). \tag{38}$$

Comparing this equation to Eq. (13) (notice $p_1 = 0$ in the sagittal plane), we get

$$W_3(\rho) = -\frac{n_i M^2(\rho)}{2s}\,. \tag{39}$$

Assuming the Abbe Sine Condition is satisfied for an optical system, then $M(\rho)$ = constant. If we also assume the image space is homogeneous, then combining equations (30), (39) and (19) leads to the following conclusions:

- When $s = t/\cos^2(\theta) = constant$, then $W_2(\rho) = 0$, $W_3(\rho) = constant$, and $\Delta x = \Delta y = 0$. This means that all the quadratic field-dependent aberrations of the system are corrected.
- When $t/\cos^2(\theta) = constant$, then $p_1{}^2 W_2(\rho) + W_3(\rho)$ is constant in the tangential plane. Therefore, $\Delta x = 0$ in the tangential plane, or all the quadratic field-dependent aberrations in the tangential plane are corrected.
- When $s = constant$, then $W_3(\rho)$ is constant. Therefore, $\Delta y = 0$ in the sagittal plane, or all the quadratic field-dependent aberrations of the form W_{2n0} (n is even) are corrected. This includes field curvature and oblique spherical aberrations.
- When $s = t/\cos^2(\theta)$, then $W_2(\rho) = 0$, and all the quadratic field-dependent aberrations of the form W_{2n2} (n is even) are corrected. The remaining quadratic field-dependent aberrations look like power or spherical aberrations for the off-axis points.

4. The Pupil Astigmatism Criteria for systems with object at infinity

When the object is at infinity, we used the Hamilton's angle characteristic to derive Pupil Astigmatism Criteria for this type of system. The form of the criteria is identical to that of the finite conjugate system (Eqs (4)-(7)), but s and t are now defined differently.

For an infinite conjugate system shown in Figure 2(b), we select an arbitrary plane which is perpendicular to the optical axis in object space. Then we trace a small cone of rays which originate from any point in the plane and are centered on the ray which is parallel to the optical axis. Again we have the tangential image at T and the sagittal image at S. We define $t = IT$ and $s = IS$. With these definitions, the Pupil Astigmatism Criteria for correcting the quadratic field-dependent aberrations for infinite conjugate systems are the same as Eqs. (4)-(7).

5. Validation of the Pupil Astigmatism Criteria

The pupil astigmatism conditions were validated in two ways, and the details of the validations are in the references publications. Particular design cases were evaluated and the actual quadratic field dependent aberrations were shown to match those predicted by the relations presented here.[3] Also, the pupil astigmatism relations, with the relevant approximations, were shown to be consistent with the accepted Seidel analysis.[5]

6. Applications in optical design

The pupil astigmatism relations above can be used to evaluate optical systems, in the same way the *offense against the sine condition* OSC can be used to determine coma. A more powerful application of these relations is for the design of new systems. By using two general aspheric surfaces, it is always possible to design an aplanatic imaging system that simultaneously satisfies Fermat's principle and the Abbe sine condition for each point in the pupil. If two more general aspherical surfaces are added, the system can be corrected for quadratic field dependent aberrations for each point in the pupil as well.[6] One such system is shown in Fig. 3, along with a plot showing the aberrations to be corrected to second order. This treatment has also been extended to plane symmetric optical systems.[7]

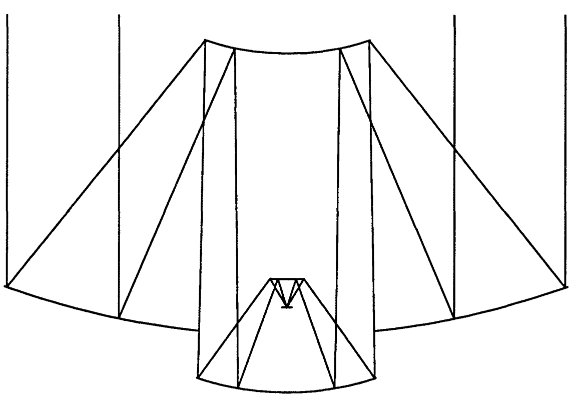


Figure 3. Layout of 4-miror telescope, fully corrected for all aberrations up to 2nd order in field.

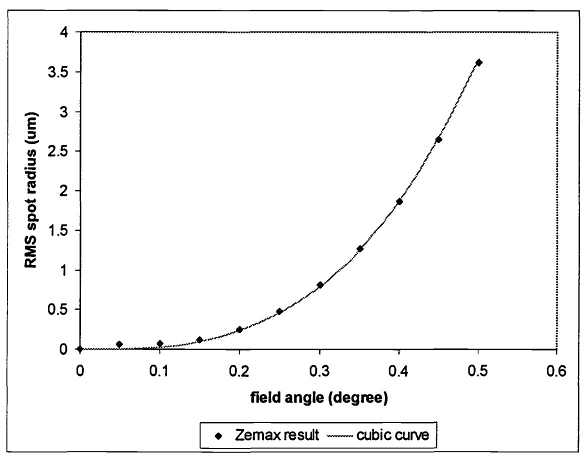


Figure 4. Field dependence of aberrations for telescope in Fig. 3.

7. Summary

In this paper we reproduce the derivation given in another paper[4], to be published in JOSA, of criteria for correcting the aberrations that have a second order field dependence and any order of pupil dependence. We name the criteria the Pupil Astigmatism Criteria due to the involvement of the astigmatism of pupil in the criteria. The criteria take identical forms for systems with the object a finite distance away and systems with the object at infinity, but the physical quantities in the criteria are defined differently for these two types of systems. The criteria involve only the properties of the rays that originate from the on-axis object point. This feature makes it very convenient to use them in optical design. Also, when the criteria are not exactly satisfied, we now have a way to calculate the residual aberrations that are quadratic in field without using any off-axis ray information. These criteria can be used together with the Fermat's Principle and the Abbe Sine Condition to design a high-NA system which performs perfectly over a moderate field of view[8].

After the first manuscript of this paper was submitted to JOSA, a reviewer pointed out that the similar set of criteria were derived by H. Boegehold and M. Herzberger[8] more than 70 years ago. They took a different approach and obtained an equivalent set of criteria for correction of all the aberrations that are quadratic in field. In this paper, we derived these criteria in a more straightforward way and presented them in a simpler form. We classify the quadratic field-dependent aberrations in 4 categories and explicitly list the condition for correction of each of them. We also derived similar conditions for systems with the object at infinity. The analysis of residual aberrations follows naturally.

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