Systematic comparison of the use of annular and Zernike circle polynomials for annular wavefronts

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The theory of wavefront analysis of a noncircular wavefront is given and applied for a systematic comparison of the use of annular and Zernike circle polynomials for the analysis of an annular wavefront. It is shown that, unlike the annular coefficients, the circle coefficients generally change as the number of polynomials used in the expansion changes. Although the wavefront fit with a certain number of circle polynomials is identically the same as that with the corresponding annular polynomials, the piston circle coefficient does not represent the mean value of the aberration function, and the sum of the squares of the other coefficients does not yield its variance. The interferometer setting errors of tip, tilt, and defocus from a four-circle-polynomial expansion are the same as those from the annular-polynomial expansion. However, if these errors are obtained from, say, an 11-circle-polynomial expansion, and are removed from the aberration function, wrong polishing will result by zeroing out the residual aberration function. If the common practice of defining the center of an interferogram and drawing a circle around it is followed, then the circle coefficients of a noncircular interferogram do not yield a correct representation of the aberration function. Moreover, in this case, some of the higher-order coefficients of aberrations that are nonexistent in the aberration function are also nonzero. Finally, the circle coefficients, however obtained, do not represent coefficients of the balanced aberrations for an annular pupil. The various results are illustrated analytically and numerically by considering an annular Seidel aberration function.

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1. Introduction

The orthonormal polynomials for an annular pupil uniquely represent balanced classical aberrations [1,2], just as the Zernike circle polynomials (hereafter known simply as circle polynomials) do for a circular pupil [1–3]. Some authors have compared the use of annular polynomials with that of the circle polynomials for annular wavefronts by considering numerical examples [4–7]. However, none has considered a systematic analysis of such a comparison.

Because the circle polynomials form a complete set, any wavefront, regardless of the shape of the pupil (which defines the perimeter of the wavefront), can be expanded in terms of them. Moreover, since each orthonormal noncircular polynomial is a linear combination of the circle polynomials, the wavefront fitting with the former set of polynomials is as good as that with the latter [8,9]. However, we illustrate the pitfalls of using circle polynomials for a noncircular pupil by considering an aberrated annular pupil. Also included is the case where the circle coefficients

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are calculated by assuming the circle polynomials to be orthogonal over an annulus. This follows the common practice of defining a center of an interferogram, drawing a unit circle around it, and determining the circle coefficients in the same manner as for a circular interferogram. The results are applied to an annular pupil aberrated by a Seidel aberration function and the annular and circle coefficients are compared. We show that not only the circle coefficients in this case yield an incorrect representation of the aberration function, but the coefficients of some of the nonexistent polynomial terms are nonzero as well.

2. Relationship Between the Orthonormal and the Corresponding Zernike Coefficients

Consider an aberration function $W(x,y)$ across a noncircular pupil. Let us fit this function with J orthonormal polynomials $F_j(x,y)$ in the form

$$\hat{W}(x,y) = \sum_{j=1}^{J} a_j F_j(x,y), \tag{1}$$

where $\hat{W}(x,y)$ is the best-fit estimate of the function with J polynomials and a_j is the coefficient of the polynomial $F_j(x,y)$. The orthonormality of the polynomials across the noncircular pupil is represented by

$$\frac{1}{A}\int_{\text{pupil}} F_j(x,y)F_{j'}(x,y)\mathrm{d}x\mathrm{d}y = \delta_{jj'}, \tag{2}$$

where $\delta_{jj'}$ is a Kronecker delta. The orthonormal coefficients are given by

$$a_j = \frac{1}{A}\int_{\text{pupil}} W(x,y)F_j(x,y)\mathrm{d}x\mathrm{d}y. \tag{3}$$

It is evident that their value does not depend on the number of polynomials J used in the expansion.

Letting $F_1(x,y) = 1$, it is easy to see from Eq. (2) that the mean value of a polynomial $F_{j\neq 1}(x,y)$ across the pupil is zero. Hence, the mean value and the variance of the estimated aberration function are given by

$$\langle \hat{W} \rangle = a_1, \tag{4}$$

$$\sigma^2_{\hat{W}} = \langle \hat{W}^2(x,y)\rangle - \langle \hat{W}(x,y)\rangle^2 \tag{5}$$

$$= \sum_{j=2}^{J} a_j^2, \tag{6}$$

where $\sigma_{\hat{W}}$ is its standard deviation. The number of polynomials J used in the expansion to estimate the aberration function is increased until $\sigma_{\hat{W}}$ approaches its true value within a certain prespecified tolerance.

Because the circle polynomials $Z_j(x,y)$ form a complete set, each orthonormal noncircular polynomial can be written as a linear sum of them in the following forms:

$$F_j(x,y) = \sum_{i=1}^{J} M_{ji} Z_i(x,y), \tag{7}$$

$$\{F_j\} = \boldsymbol{M}\{Z_j\}, \tag{8}$$

where M_{ji} are the elements of the lower triangular conversion matrix $\boldsymbol{M}$. The estimated aberration function can accordingly be expanded in terms of the circle polynomials in the form

$$\hat{W}(x,y) = \sum_{j=1}^{J} \hat{b}_j Z_j(x,y), \tag{9}$$

where $\hat{b}_j$ is the Zernike coefficient of a polynomial $Z_j(x,y)$. The circle polynomials are orthonormal in Cartesian coordinates across a unit circle according to

$$\frac{1}{\pi}\int_{x^2+y^2\leq 1} Z_j(x,y)Z_{j'}(x,y)\mathrm{d}x\mathrm{d}y = \delta_{jj'}, \tag{10a}$$

or in polar coordinates (with $x = \rho\cos\theta$ and $y = \rho\sin\theta$):

$$\frac{1}{\pi}\int_0^1\int_0^{2\pi} Z_j(\rho,\theta)Z_{j'}(\rho,\theta)\rho\mathrm{d}\rho\mathrm{d}\theta = \delta_{jj'}. \tag{10b}$$

Substituting Eq. (7) into Eq. (1), we obtain

$$\hat{W}(x,y) = \sum_{j=1}^{J} a_j \sum_{i=1}^{j} M_{ji} Z_i(x,y) = \sum_{j=1}^{J}\sum_{i=j}^{J} a_i M_{ij} Z_j(x,y). \tag{11}$$

Comparing Eqs. (9) and (11), we obtain

$$\hat{b}_j = \sum_{i=j}^{J} a_i M_{ij}. \tag{12}$$

Evidently, the value of a coefficient $\hat{b}_j$ depends on the number of polynomials J used in the expansion. Equation (12) can be written in a matrix form as

$$\hat{\boldsymbol{b}} = \boldsymbol{M}^T \boldsymbol{a}, \tag{13}$$

where $\boldsymbol{a}$ and $\hat{\boldsymbol{b}}$ are the column vectors representing the orthonormal and the Zernike coefficients, respectively, and $\boldsymbol{M}^T$ is the transpose of the conversion matrix $\boldsymbol{M}$. Thus, the matrix that is used to obtain the orthonormal polynomials from the circle polynomials is also used to obtain the Zernike coefficients from the orthonormal coefficients. The transpose of a matrix is obtained by interchanging its rows and columns. Because $\boldsymbol{M}$ is a lower triangular matrix, $\boldsymbol{M}^T$ is an upper triangular matrix. Multiplying both sides of Eq. (13) by the inverse $(\boldsymbol{M}^T)^{-1}$ of $\boldsymbol{M}^T$, we obtain

$$\boldsymbol{a} = (\boldsymbol{M}^T)^{-1}\hat{\boldsymbol{b}}. \tag{14}$$

Accordingly, if the Zernike coefficients are known, the orthonormal coefficients can be obtained from them. It should be evident from Eq. (12) or Eq. (13) that a Zernike coefficient is a linear combination of the orthonormal coefficients, just as an orthonormal polynomial is a linear combination of the circle polynomials.

If the orthonormal coefficients are not known, the Zernike coefficients $\hat{b}_j$ can be obtained by a least-squares fit. Suppose the aberration values are known over a certain domain by way of interferometry at N data points. Equation (9) can be written in matrix form

$$\hat{S} = \boldsymbol{Z}\hat{\boldsymbol{b}}, \tag{15}$$

where $\hat{S}$ is an array of N elements representing the values of the aberration function $\hat{W}(x,y)$, and $\boldsymbol{Z}$ is an $N \times J$ matrix representing each of the J polynomials over the N data points. Solving Eq. (15), for example, with a standard singular-value decomposition algorithm yields

$$\hat{\boldsymbol{b}} = \boldsymbol{Z}^{-1}\hat{S}, \tag{16}$$

where $\boldsymbol{Z}^{-1}$ is a generalized inverse of the $\boldsymbol{Z}$ matrix. Of course, this procedure can also be used to determine the orthonormal coefficients by replacing the circle polynomials with the orthonormal polynomials. Except for any numerical error because of the finite number N of the data points, the $\hat{b}$ coefficients given by Eq. (16) are the same as those given by Eq. (13).

If the practice of drawing a unit circle around an interferogram and determining the Zernike coefficients for a circular pupil is extended to a noncircular wavefront, the coefficients thus obtained will be given by

$$b_j = \frac{1}{A}\int\limits_{\text{pupil}} W(x,y)Z_j(x,y)\mathrm{d}x\mathrm{d}y. \tag{17}$$

The circle polynomials in Eq. (17) are implicitly assumed to be orthonormal over the noncircular pupil. The value of a circle coefficient b_j does not depend on the number of polynomials used in the expansion. Substituting Eq. (1) for the estimated aberration function $\hat{W}(x,y)$ in terms of the orthonormal polynomials, we obtain

$$b_j = \sum_{j'=1}^{J} a_{j'}\frac{1}{A}\int\limits_{\text{pupil}} Z_j(x,y)F_{j'}(x,y)\mathrm{d}x\mathrm{d}y = \sum_{j'=1}^{J} a_{j'}\langle Z_j|F_{j'}\rangle, \tag{18}$$

or in a matrix form

$$\boldsymbol{b} = \boldsymbol{C}^{ZF}\boldsymbol{a}, \tag{19}$$

where $\boldsymbol{C}^{ZF}$ is a matrix representing the inner products $\langle Z_j|F_{j'}\rangle$ of the Zernike polynomials with the orthonormal polynomials over the domain of the noncircular wavefront. As illustrated in Section 7, by considering a Seidel aberration function, the Zernike coefficients b_j thus obtained are incorrect in the sense that they do not yield a least-squares fit of the aberration function $W(x,y)$, unlike the coefficients $\hat{b}_j$. This, of course, is due to the incorrect assumption of orthonormality of the circle polynomials over the noncircular pupil.

To relate the $\hat{b}$ and the b coefficients, we equate the right-hand sides of Eqs. (1) and (9), multiply both sides by $Z_{j'}$, and integrate over the domain of the noncircular pupil. Thus,

$$\sum_{j=1}^{J}\hat{b}_j Z_j(x,y) = \sum_{j=1}^{J} a_j F_j(x,y), \tag{20}$$

$$\sum_{j=1}^{J}\hat{b}_j\langle Z_{j'}|Z_j\rangle = \sum_{j=1}^{J} a_j\langle Z_{j'}|F_j\rangle, \tag{21}$$

$$\boldsymbol{C}^{ZZ}\hat{\boldsymbol{b}} = \boldsymbol{C}^{ZF}\boldsymbol{a} = \boldsymbol{b}, \tag{22}$$

where we have utilized Eq. (19). From Eqs. (13) and (22), it is evident that

$$\boldsymbol{C}^{ZF} = \boldsymbol{C}^{ZZ}\boldsymbol{M}^T. \tag{23}$$

Typical elements of the matrices $\boldsymbol{C}^{ZZ}$ and $\boldsymbol{C}^{ZF}$ are given by

$$c_{jj'} = \frac{1}{A}\int\limits_{\text{pupil}} Z_j(x,y)Z_{j'}(x,y)\mathrm{d}x\mathrm{d}y, \tag{24}$$

$$d_{jj'} = \frac{1}{A}\int\limits_{\text{pupil}} Z_j(x,y)F_{j'}(x,y)\mathrm{d}x\mathrm{d}y. \tag{25}$$

It is evident from Eq. (24) that $c_{jj'} = c_{j'j}$.

3. Orthonormal Annular Polynomials

Consider a system with a unit annular pupil with an obscuration ratio. Thus the inner and outer radii of the pupil are ϵ and unity. The polynomials $A_j(\rho,\theta;\epsilon)$ that are orthonormal across it and represent balanced aberrations for it can be obtained from the circle polynomials by the Gram–Schmidt orthogonalization process [1,2]. Like a circle polynomial, an annular polynomial is also separable in its dependence on the radial coordinate ρ and the azimuthal angle θ. The dependence on the obscuration ratio $\boldsymbol{\epsilon}$ is contained only in the radial portion of the polynomial. The polynomials are given by

$$A_{\mathrm{even}j}(\rho,\theta;\boldsymbol{\epsilon}) = \sqrt{2(n+1)}R_n^m(\rho;\boldsymbol{\epsilon})\cos m\theta, \qquad m \neq 0, \tag{26a}$$

$$A_{\mathrm{odd}j}(\rho,\theta;\boldsymbol{\epsilon}) = \sqrt{2(n+1)}R_n^m(\rho;\boldsymbol{\epsilon})\sin m\theta, \qquad m \neq 0, \tag{26b}$$

$$A_j(\rho,\theta;\boldsymbol{\epsilon}) = \sqrt{n+1}R_n^0(\rho;\boldsymbol{\epsilon}), \qquad m = 0, \tag{26c}$$

where $\boldsymbol{\epsilon} \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$, n and m are positive integers, and $n - m \geq 0$ and positive. Except for the normalization constant, an annular polynomial with $n = m$ has the same form as that for a corresponding circle polynomial. The index j is a polynomial ordering number in the same manner as it is for the circle polynomials. Thus, the polynomials are ordered such that an even j corresponds to a symmetric polynomial varying as $\cos m\theta$, while an odd j corresponds to an antisymmetric polynomial varying as $\sin m\theta$. A polynomial with a lower value of n is ordered first, and for a given value of n, a polynomial with a lower value of m is ordered first. The annular polynomials are orthonormal across the annular pupil according to

$$\frac{1}{\pi(1-\boldsymbol{\epsilon}^2)}\int_{\boldsymbol{\epsilon}}^{1}\int_{0}^{2\pi} A_j(\rho,\theta;\boldsymbol{\epsilon})A_{j'}(\rho,\theta;\boldsymbol{\epsilon})\rho\,\mathrm{d}\rho\,\mathrm{d}\theta = \delta_{jj'}. \tag{27}$$

As $\boldsymbol{\epsilon} \to 0$, an annular polynomial reduces to a corresponding circle polynomial and Eq. (27) reduces to Eq. (10b).

The annular polynomials can be written in terms of the circle polynomials according to

$$\{A_j\} = \boldsymbol{M}\{Z_j\}, \tag{28}$$

where $\boldsymbol{M}$ is the conversion matrix. An annular aberration function $W(\rho,\theta;\boldsymbol{\epsilon})$ can be estimated by J orthonormal polynomials according to

$$\hat{W}(\rho,\theta;\boldsymbol{\epsilon}) = \sum_{j=1}^{J} a_j A_j(\rho,\theta;\boldsymbol{\epsilon}), \tag{29}$$

where the orthonormal annular expansion coefficients are given by

$$a_j = \frac{1}{\pi(1-\boldsymbol{\epsilon}^2)}\int_{\boldsymbol{\epsilon}}^{1}\int_{0}^{2\pi} W(\rho,\theta;\boldsymbol{\epsilon})A_j(\rho,\theta;\boldsymbol{\epsilon})\rho\,\mathrm{d}\rho\,\mathrm{d}\theta. \tag{30}$$

The mean value and the variance of the estimated function are accordingly given by Eqs. (4) and (6).

Table 1 lists the first 11 annular polynomials along with the names associated with some of them. They are given in terms of the circle polynomials in Table 2. The nonzero elements of an 11×11 conversion matrix, as obtained from Table 2, are listed in Table 3. The transpose matrix $\boldsymbol{M}^T$ can be obtained easily by interchanging the rows and columns of $\boldsymbol{M}$. The nonzero elements of the 11×11 matrices $\boldsymbol{C}^{ZZ}$ and $\boldsymbol{C}^{ZF}$ obtained from Eqs. (24) and (25) by integrating over an annular pupil and replacing $F_{j'}(x,y)$ by $A_{j'}(\rho,\theta;\epsilon)$ are given in Tables 4 and 5, respectively.

4. Annular and Zernike Circle Coefficients of an Annular Wavefront

Given a certain annular aberration function, its annular coefficients can be determined from Eq. (30). If it is expanded in terms of only the first four circle polynomials, i.e., if $J = 4$, then the expansion $\hat{b}$ coefficients according to Eq. (13) are given by

$$\begin{pmatrix}\hat{b}_1\\ \hat{b}_2\\ \hat{b}_3\\ \hat{b}_4\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -\sqrt{3}\boldsymbol{\epsilon}^2(1-\boldsymbol{\epsilon}^2)^{-1} \\ 0 & (1+\boldsymbol{\epsilon}^2)^{-1/2} & 0 & 0 \\ 0 & 0 & (1+\boldsymbol{\epsilon}^2)^{-1/2} & 0 \\ 0 & 0 & 0 & (1-\boldsymbol{\epsilon}^2)^{-1} \end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ a_3\\ a_4\end{pmatrix} = \begin{pmatrix} a_1 - \sqrt{3}\boldsymbol{\epsilon}^2(1-\boldsymbol{\epsilon}^2)^{-1}a_4 \\ (1+\boldsymbol{\epsilon}^2)^{-1/2}a_2 \\ (1+\boldsymbol{\epsilon}^2)^{-1/2}a_3 \\ (1-\boldsymbol{\epsilon}^2)^{-1}a_4 \end{pmatrix}, \tag{31}$$

$$\hat{b}_1 = a_1 - \sqrt{3}\boldsymbol{\epsilon}^2(1-\boldsymbol{\epsilon}^2)^{-1}a_4, \tag{32a}$$

$$\hat{b}_2 = (1+\boldsymbol{\epsilon}^2)^{-1/2}a_2, \tag{32b}$$

$$\hat{b}_3 = (1+\boldsymbol{\epsilon}^2)^{-1/2}a_3, \tag{32c}$$

$$\hat{b}_4 = (1-\boldsymbol{\epsilon}^2)^{-1}a_4. \tag{32d}$$

These coefficients represent the Zernike piston, tip, tilt, and defocus coefficients, respectively.

To see how these coefficients change as more polynomials are used in the expansion, we consider an expansion using 11 circle polynomials.

Table 1. Orthonormal Annular Polynomials $A_j(\rho,\theta;\epsilon)$ for an Obscuration Ratio ϵ

j	n	m	$A_j(\rho,\theta;\epsilon)$	Aberration Name
1	0	0	1	Piston
2	1	1	$2[\rho/(1+\epsilon^2)^{1/2}]\cos\theta$	x tilt
3	1	1	$2[\rho/(1+\epsilon^2)^{1/2}]\sin\theta$	y tilt
4	2	0	$\sqrt{3}(2\rho^2-1-\epsilon^2)/(1-\epsilon^2)$	Defocus
5	2	2	$\sqrt{6}[\rho^2/(1+\epsilon^2+\epsilon^4)^{1/2}]\sin 2\theta$	45° Primary astigmatism
6	2	2	$\sqrt{6}[\rho^2/(1+\epsilon^2+\epsilon^4)^{1/2}]\cos 2\theta$	0° Primary astigmatism
7	3	1	$\sqrt{8}\frac{3(1+\epsilon^2)\rho^3-2(1+\epsilon^2+\epsilon^4)\rho}{(1-\epsilon^2)[(1+\epsilon^2)(1+4\epsilon^2+\epsilon^4)]^{1/2}}\sin\theta$	Primary y coma
8	3	1	$\sqrt{8}\frac{3(1+\epsilon^2)\rho^3-2(1+\epsilon^2+\epsilon^4)\rho}{(1-\epsilon^2)[(1+\epsilon^2)(1+4\epsilon^2+\epsilon^4)]^{1/2}}\cos\theta$	Primary x coma
9	3	3	$\sqrt{8}[\rho^3/(1+\epsilon^2+\epsilon^4+\epsilon^6)^{1/2}]\sin 3\theta$	
10	3	3	$\sqrt{8}[\rho^3/(1+\epsilon^2+\epsilon^4+\epsilon^6)^{1/2}]\cos 3\theta$	
11	4	0	$\sqrt{5}[6\rho^4-6(1+\epsilon^2)\rho^2+1+4\epsilon^2+\epsilon^4]/(1-\epsilon^2)^2$	Primary spherical aberration

The coefficients are now given by

$$\hat{b}_1 = a_1 - 3\epsilon^2(1-\epsilon^2)^{-1}a_4 + \sqrt{5}\epsilon^2(1+\epsilon^2)(1-\epsilon^2)^{-2}a_{11}, \tag{33a}$$

$$\hat{b}_2 = (1+\epsilon^2)^{-1/2}a_2 - \left(2\sqrt{2}\,\epsilon^4/B\right)a_8, \tag{33b}$$

$$\hat{b}_3 = (1+\epsilon^2)^{-1/2}a_3 - \left(2\sqrt{2}\,\epsilon^4/B\right)a_7, \tag{33c}$$

$$\hat{b}_4 = (1-\epsilon^2)^{-1}a_4 - \sqrt{15}\epsilon^2(1-\epsilon^2)^{-2}a_{11}, \tag{33d}$$

$$\hat{b}_5 = (1+\epsilon^2+\epsilon^4)^{-1/2}a_5, \tag{33e}$$

$$\hat{b}_6 = (1+\epsilon^2+\epsilon^4)^{-1/2}a_6, \tag{33f}$$

$$\hat{b}_7 = [(1+\epsilon^2)/B]a_7, \tag{33g}$$

$$\hat{b}_8 = [(1+\epsilon^2)/B]a_8, \tag{33h}$$

$$\hat{b}_9 = (1+\epsilon^2+\epsilon^4+\epsilon^6)^{-1/2}a_9, \tag{33i}$$

$$\hat{b}_{10} = (1+\epsilon^2+\epsilon^4+\epsilon^6)^{-1/2}a_{10}, \tag{33j}$$

$$\hat{b}_{11} = (1-\epsilon^2)^{-2}a_{11}, \tag{33k}$$

where

$$B = (1-\epsilon^2)[(1+\epsilon^2)(1+4\epsilon^2+\epsilon^4)]^{1/2}. \tag{34}$$

It is evident that all of the first four coefficients change, and $b_j = M_{jj}a_j$ for $5 \le j \le 11$. The Zernike astigmatism coefficients $\hat{b}_5$ and $\hat{b}_6$ are smaller than the corresponding annular coefficients a_5 and a_6 by a factor of $(1+\epsilon^2+\epsilon^4)^{1/2}$. However, the Zernike spherical aberration coefficient $\hat{b}_{11}$ is larger than the corresponding annular coefficient a_{11} by a factor of $(1-\epsilon^2)^{-2}$. For example, when $\epsilon = 0.5$, the astigmatism coefficients are smaller by a factor of 1.1456 and the spherical aberration coefficient is larger by a factor of 1.7778.

It should be evident that, because of the orthogonality of the trigonometric functions, there is correlation between an annular and a circle polynomial only if they have the same azimuthal dependence. As a consequence, the piston coefficient $\hat{b}_1$, for example, is a linear combination of the piston coefficient a_1, defocus coefficient a_4, and various orders of spherical aberration. Similarly, the tilt coefficient $\hat{b}_2$ is a linear combination of the tilt coefficient a_2 and various orders of coma, or astigmatism coefficient $\hat{b}_5$ is a linear combination of various orders of astigmatism. Accordingly, the astigmatism coefficients change if a 13-polynomial expansion is considered. For example, $\hat{b}_5$ then contains contribution from a_{13} as well. The tip and tilt coefficients $\hat{b}_2$ and $\hat{b}_3$ change further if polynomials A_{16} (varying as $\cos\theta$) and A_{17} (varying as $\sin\theta$) are included in the expansion. Moreover, A_{16} also contributes to the coma coefficient $\hat{b}_8$, and A_{17} similarly contributes to the coma coefficient $\hat{b}_7$. The defocus coefficient $\hat{b}_4$ does not change until the secondary spherical aberration polynomial A_{22} is included with its coefficient a_{22}. Its inclusion also affects the primary spherical aberration coefficient $\hat{b}_{11}$. Thus, it is easy to see which, when, and by how much the $\hat{b}_j$ coefficients change, depending on the number of polynomials used in the expansion.

We note that the mean value of the aberration function is given by the annular piston coefficient a_1. However, the value of the corresponding Zernike

Table 2. Annular Polynomials $A_j(\rho,\theta;\epsilon)$ in Terms of the Zernike Circle Polynomials $Z_j(\rho,\theta)$[a]

$$A_1 = Z_1$$
$$A_2 = (1+\epsilon^2)^{-1/2} Z_2$$
$$A_3 = (1+\epsilon^2)^{-1/2} Z_3$$
$$A_4 = (1-\epsilon^2)^{-1}(-\sqrt{3}\,\epsilon^2 Z_1 + Z_4)$$
$$A_5 = (1+\epsilon^2+\epsilon^4)^{-1/2} Z_5$$
$$A_6 = (1+\epsilon^2+\epsilon^4)^{-1/2} Z_6$$
$$A_7 = B^{-1}[-2\sqrt{2}\epsilon^4 Z_3 + (1+\epsilon^2) Z_7]$$
$$A_8 = B^{-1}[-2\sqrt{2}\epsilon^4 Z_2 + (1+\epsilon^2) Z_8]$$
$$A_9 = (1+\epsilon^2+\epsilon^4+\epsilon^6)^{-1/2} Z_9$$
$$A_{10} = (1+\epsilon^2+\epsilon^4+\epsilon^6)^{-1/2} Z_{10}$$
$$A_{11} = (1-\epsilon^2)^{-2}[\sqrt{5}\epsilon^2(1+\epsilon^2) Z_1 - \sqrt{15}\epsilon^2 Z_4 + Z_{11}]$$
$$B = (1-\epsilon^2)[(1+\epsilon^2)(1+4\epsilon^2+\epsilon^4)]^{1/2}$$

[a]Where ϵ is the obscuration ratio of the annular pupil.

circle coefficient $\hat{b}_1$ depends on the number of polynomials used in the expansion, and it does not equal a_1 and, therefore, does not represent the mean value. An orthonormal annular coefficient (other than piston) represents the standard deviation of the corresponding aberration term in the expansion, but a Zernike circle coefficient generally does not. The variance of the aberration function cannot be obtained by summing the squares of the Zernike circle coefficients $\hat{b}_j$s (excluding the piston coefficient). The circle coefficients b_js can be obtained from the $\hat{b}_j$ or the a_j coefficients according to Eq. (22). They are considered in Section 7 for a Seidel aberration function.

5. Interferometer Setting Errors

The estimated wavefront obtained by using only the first four polynomials represents the best-fit parabolic approximation of the aberration function in a least-squares sense. In terms of the orthonormal annular polynomials, it can be written as

$$\hat{W}(x,y) = a_1 A_1 + a_2 A_2 + a_3 A_3 + a_4 A_4 \tag{35a}$$

$$= a_1 + 2(1+\epsilon^2)^{-1/2} a_2 x + 2(1+\epsilon^2)^{-1/2} a_3 y + \sqrt{3}(1-\epsilon^2)^{-1} a_4[-\epsilon^2 + (2\rho^2 - 1)]. \tag{35b}$$

Table 3. Nonzero Elements of 11×11 Conversion Matrix M for Obtaining the Annular Polynomials $A_j(\rho,\theta;\epsilon)$ from the Zernike Circle Polynomials $Z_j(\rho,\theta)$

$$M_{11} = 1$$
$$M_{22} = (1+\epsilon^2)^{-1/2} = M_{33}$$
$$M_{41} = -\sqrt{3}\epsilon^2(1-\epsilon^2)^{-1}$$
$$M_{44} = (1-\epsilon^2)^{-1}$$
$$M_{55} = (1+\epsilon^2+\epsilon^4)^{-1/2} = M_{66}$$
$$M_{73} = -2\sqrt{2}\epsilon^4 B^{-1} = M_{82}$$
$$M_{77} = (1+\epsilon^2) B^{-1} = M_{88}$$
$$M_{99} = (1+\epsilon^2+\epsilon^4+\epsilon^6)^{-1/2} = M_{10,10}$$
$$M_{11,1} = \sqrt{5}\epsilon^2(1+\epsilon^2)(1-\epsilon^2)^{-2}$$
$$M_{11,4} = -\sqrt{15}\epsilon^2(1-\epsilon^2)^{-2}$$
$$M_{11,11} = (1-\epsilon^2)^{-2}$$

Table 4. Nonzero Elements $c_{jj'}$ of 11×11 Matrix C^{ZZ} of the Zernike Circle Polynomials over an Annular Pupil of Obscuration Ratio ϵ, where $c_{jj'} = c_{j'j}$

$$c_{11} = 1$$
$$c_{14} = \sqrt{3}\epsilon^2 = c_{41}$$
$$c_{1,11} = -\sqrt{5}\epsilon^2(1-2\epsilon^2) = c_{11,1}$$
$$c_{22} = 1+\epsilon^2 = c_{33}$$
$$c_{28} = 2\sqrt{2}\epsilon^4 = c_{82} = c_{37} = c_{73}$$
$$c_{44} = 1 - 2\epsilon^2 + 4\epsilon^4$$
$$c_{4,11} = \sqrt{15}\epsilon^2(1 - 3\epsilon^2 + 3\epsilon^4) = c_{11,4}$$
$$c_{55} = 1+\epsilon^2+\epsilon^4 = c_{66}$$
$$c_{77} = 1+\epsilon^2 - 7\epsilon^4 + 9\epsilon^6 = c_{88}$$
$$c_{99} = 1+\epsilon^2+\epsilon^4+\epsilon^6 = c_{10,10}$$
$$c_{11,11} = 1 - 4\epsilon^2 + 26\epsilon^4 - 54\epsilon^6 + 36\epsilon^8$$

In terms of the circle polynomials, it can be written

$$\hat{W}(x,y) = \hat{b}_1 Z_1 + \hat{b}_2 Z_2 + \hat{b}_3 Z_3 + \hat{b}_4 Z_4 \tag{36a}$$

$$= \hat{b}_1 + 2\hat{b}_2 x + 2\hat{b}_3 y + \sqrt{3}\hat{b}_4(2\rho^2 - 1). \tag{36b}$$

In Eqs. (35) and (36), we have omitted the arguments of the annular and circle polynomials for simplicity. Substituting for the $\hat{b}_j$ coefficients from Eqs. (32), we find that the coefficients of x, y, and ρ^2 representing the tip, tilt, and defocus values obtained from the Zernike coefficients, respectively, are the same as those obtained from the orthonormal coefficients. The estimated piston from the Zernike expansion of Eq. (36b) is $\hat{b}_1 - \sqrt{3}\hat{b}_4$, which is the same as $a_1 - \sqrt{3}\epsilon^2(1-\epsilon^2)^{-1} a_4$ from the orthonormal expansion of Eq. (35b). Accordingly, the aberration function obtained by subtracting the piston, tip, tilt, and defocus values from the measured aberration function is independent of the nature of the polynomials used in the expansion, so long as the nonorthogonal expansion is in terms of only the first four circle polynomials [as may be seen, for example, by comparing Eqs. (33a)–(33d) with Eqs. (32a)–(32d)]. In an interferometer, the tip and tilt represent the lateral errors and defocus represents the longitudinal error in the location of a point source illuminating an optical surface under test from its center of curvature. These four terms are generally removed from the aberra-

Table 5. Nonzero Elements $d_{jj'}$ of 11×11 Matrix C^{ZF} of the Zernike Circle Polynomials over an Annular Pupil of Obscuration Ratio ϵ

$$d_{11} = 1$$
$$d_{22} = (1+\epsilon^2)^{1/2} = d_{33}$$
$$d_{41} = \sqrt{3}\epsilon^2$$
$$d_{44} = (1 - 2\epsilon^2 + \epsilon^4)(1-\epsilon^2)^{-1}$$
$$d_{55} = (1+\epsilon^2+\epsilon^4)^{1/2} = d_{66}$$
$$d_{73} = 2\sqrt{2}\epsilon^4(1+\epsilon^2)^{-1/2} = d_{82}$$
$$d_{77} = (1-\epsilon^2)(1+4\epsilon^2+\epsilon^4)^{1/2}(1+\epsilon^2)^{-1/2} = d_{88}$$
$$d_{99} = (1+\epsilon^2+\epsilon^4+\epsilon^6)^{1/2} = d_{10,10}$$
$$d_{11,1} = -\sqrt{5}\epsilon^2(1-2\epsilon^2)$$
$$d_{11,4} = \sqrt{15}\epsilon^2(1-\epsilon^2)$$
$$d_{11,11} = (1-\epsilon^2)^2$$

tion function and the remaining function is given to the optician to zero out from the optical surface by polishing.

6. Wavefront Fitting

When an aberration function is expanded in terms of the orthonormal polynomials, one or more polynomial terms can be added or subtracted from the aberration function without affecting the coefficients of the other polynomials in the expansion. However, that is generally not true with the Zernike expansion. This is due to the fact that an expansion in terms of the orthonormal polynomials gives a best fit for each polynomial, but an expansion in terms of the circle polynomials gives it for the whole set in the expansion. The estimated or reconstructed wavefront by the same number of corresponding orthonormal or Zernike polynomials is the same. For example, the four-polynomial aberration functions of Eqs. (35) and (36) are identically the same function.

Although the wavefront fit with a certain number of circle polynomials is as good as the fit with a corresponding set of the orthonormal polynomials, there are pitfalls in using the circle polynomials. Because the circle polynomials are not orthogonal over the noncircular pupil, the advantages of orthogonality and aberration balancing are lost. Because they do not represent the balanced classical aberrations for a noncircular pupil, the Zernike coefficients $\hat{b}_j$s do not have the physical significance of their orthonormal counterparts. For example, the mean value of a circle polynomial across a noncircular pupil is not zero, the Zernike piston coefficient does not represent the mean value of the aberration, the other Zernike coefficients do not represent the standard deviation of the corresponding aberration terms, and the variance of the aberration is not equal to the sum of the squares of these other coefficients. Moreover, the value of a Zernike coefficient generally changes as the number of polynomials used in the expansion of an aberration function changes. Hence, the circle polynomials are not appropriate for the analysis of a noncircular wavefront. Of course, wavefront fitting with the improperly calculated Zernike coefficients b_j by using Eq. (17) will be in error, as demonstrated in Section 7 for a Seidel aberration function.

7. Application: Annular and Zernike Circle Coefficients of an Annular Seidel Aberration Function

Consider an annular pupil aberrated by a Seidel aberration function given by

$$W(\rho,\theta;\epsilon) = A_t\rho\cos\theta + A_d\rho^2 + A_a\rho^2\cos^2\theta + A_c\rho^3\cos\theta + A_s\rho^4, \qquad \epsilon \le \rho \le 1, \quad (37)$$

where A_t, A_d, A_a, A_c, and A_s represent the peak values of distortion, field curvature, astigmatism, coma, and spherical aberration, respectively. Without the explicit field dependence, distortion is equivalent to a wavefront tilt, and field curvature is equivalent to a wavefront defocus. The aberration function when approximated by only four annular polynomials can be written

$$\hat{W}(\rho,\theta;\epsilon) = a_1A_1 + a_2A_2 + a_4A_4, \quad (38)$$

where the expansion coefficients according to Eq. (30) are given by

$$a_1 = (1+\epsilon^2)(2A_d + A_a)/4 + (1+\epsilon^2+\epsilon^4)A_s/3, \quad (39a)$$

$$a_2 = (1+\epsilon^2)^{1/2}A_t/2 + (1+\epsilon^2+\epsilon^4)(1+\epsilon^2)^{-1/2}A_c/3, \quad (39b)$$

$$a_4 = (1-\epsilon^2)(2A_d + A_a)/4\sqrt{3} + (1-\epsilon^4)A_s/2\sqrt{3}. \quad (39c)$$

It should be evident that the coefficient a_3 of the annular polynomial A_3 varying as $\sin\theta$ is zero. The mean value of the estimated aberration function is given by a_1, and its variance is given by

$$\sigma^2_{\hat{W}} = a_2^2 + a_4^2. \quad (40)$$

An expansion in terms of 11 annular polynomials can be written

$$W(\rho,\theta;\epsilon) = a_1A_1 + a_2A_2 + a_4A_4 + a_6A_6 + a_8A_8 + a_{11}A_{11}, \quad (41)$$

where the coefficients a_1, a_2, and a_4 are given by Eqs. (39a)–(39c) and

$$a_6 = \frac{1}{2\sqrt{6}}(1+\epsilon^2+\epsilon^4)^{1/2}A_a, \quad (39d)$$

$$a_8 = \frac{1-\epsilon^2}{6\sqrt{2}}\left(\frac{1+4\epsilon^2+\epsilon^4}{1+\epsilon^2}\right)^{1/2}A_c, \quad (39e)$$

$$a_{11} = \frac{(1-\epsilon^2)^2}{6\sqrt{5}}A_s. \quad (39f)$$

Again, it should be evident that the coefficients a_5, a_7, and a_9 of the polynomials A_5, A_7, and A_9, respectively, varying as $\sin m\theta$ are zero. Moreover, the coefficient a_{10} of the polynomial A_{10} varying as $\cos 3\theta$ is also zero. The 11-polynomial expansion represents the Seidel aberration function exactly. Its mean value is again a_1, as given by Eq. (39a), and its variance is given by

$$\sigma^2_W = a_2^2 + a_4^2 + a_6^2 + a_8^2 + a_{11}^2. \quad (42)$$

Next, we expand the Seidel aberration function in terms of the circle polynomials. A four-polynomial expansion can be obtained from Eqs. (32) and (39) in the form

$$\hat{W}(\rho, \theta; \epsilon) = \hat{b}_1 Z_1 + \hat{b}_2 Z_2 + \hat{b}_4 Z_4, \tag{43}$$

where

$$\hat{b}_1 = (2A_d + A_a)/4 + [1 - \epsilon^2(1+\epsilon^2)/2]A_s/3, \tag{44a}$$

$$\hat{b}_2 = a_2/(1+\epsilon^2)^{1/2}, \tag{44b}$$

$$\hat{b}_4 = a_4/(1-\epsilon^2). \tag{44c}$$

The estimated aberration function in Eq. (43) is exactly the same as that in Eq. (38), and the values of piston, x tilt, and defocus are exactly the same as those obtained from Eqs. (39a)–(39c). It should be evident, however, that its mean value is not given by $\hat{b}_1$. Moreover, because an expansion coefficient does not represent the standard deviation of the corresponding aberration polynomial term, its variance is not given by $\hat{b}_2^2 + \hat{b}_4^2$.

From Eqs. (33) and (39), an 11-polynomial expansion can be written:

$$W(\rho, \theta; \epsilon) = \hat{b}_1 Z_1 + \hat{b}_2 Z_2 + \hat{b}_4 Z_4 + \hat{b}_6 Z_6 + \hat{b}_8 Z_8 + \hat{b}_{11} Z_{11}, \tag{45}$$

where

$$\hat{b}_1 = (2A_d + A_a)/4 + A_s/3, \tag{46a}$$

$$\hat{b}_2 = A_t/2 + A_c/3, \tag{46b}$$

$$\hat{b}_4 = (2A_d + A_a)/4\sqrt{3} + A_s/2\sqrt{3}, \tag{46c}$$

$$\hat{b}_6 = A_a/2\sqrt{6}, \tag{46d}$$

$$\hat{b}_8 = A_c/6\sqrt{2}, \tag{46e}$$

$$\hat{b}_{11} = A_s/6\sqrt{5}. \tag{46f}$$

As in the case of annular polynomials, the 11 circle polynomials also represent the Seidel aberration function exactly. The expansion coefficients can also be obtained by inspection of the aberration function and the form of the circle polynomials. Indeed, because of the form of the Seidel aberration function, the circle coefficients are independent of the obscuration ratio ϵ. Each $\hat{b}$ coefficient represents the value of the corresponding a coefficient for $\epsilon = 0$. It is clear that each of the three nonzero coefficients of the four-polynomial expansion changes as the number of polynomials is increased from 4 to 11. Hence, the values of piston, x tilt, and defocus obtained from the coefficients $\hat{b}_1$, $\hat{b}_2$, and $\hat{b}_4$ are incorrect. Again, the mean value of the aberration function is not given by $\hat{b}_1$ and its variance is not given by the sum of the squares of the other coefficients.

If we consider the first four polynomial terms as representing the interferometer setting errors and remove them from the aberration function, the residual aberration function from the annular expansion is given by

$$W_{\mathrm{RA}}(\rho, \theta; \epsilon) = a_6 A_6 + a_8 A_8 + a_{11} A_{11}. \tag{47}$$

The same residual aberration function is obtained if a four-polynomial Zernike expansion of Eq. (43) is subtracted from the aberration function $W(\rho, \theta; \epsilon)$. However, if the first four polynomials are subtracted from the aberration function of Eq. (45), the residual aberration function is given by

$$\begin{aligned} W_{\mathrm{RC}\hat{b}}(\rho, \theta; \epsilon) &= \hat{b}_6 Z_6 + \hat{b}_8 Z_8 + \hat{b}_{11} Z_{11} \\ &= (A_a/2\sqrt{6})Z_6 + (A_c/6\sqrt{2})Z_8 \\ &\quad + (A_s/6\sqrt{5})Z_{11}. \end{aligned} \tag{48}$$

Because the 11-polynomial aberration functions of Eqs. (41) and (45) are equal to each other [and equal to the Seidel aberration function of Eq. (37)], the difference between the residual aberration functions of Eqs. (48) and (47) is equal to the difference between the interferometer setting errors given by Eq. (38) or (43) and those given by Eq. (45). Accordingly, the difference or the error function consists of piston, tilt, and defocus only. It is given by

$$\begin{aligned} \Delta W_{R\hat{b}}(\rho, \theta; \epsilon) &= -\frac{1}{6}\epsilon^2(4+\epsilon^2)A_s + \frac{2}{3}\frac{\epsilon^4}{1+\epsilon^2}A_c \rho \cos\theta \\ &\quad + \epsilon^2 A_s \rho^2, \end{aligned} \tag{49}$$

and is independent of the number J of the annular and circle polynomials (e.g., 11, as above) used in the expansion. Of course, piston does not affect the peak-to-valley value or the variance of the aberration function. If the interferometer setting errors obtained from Eq. (45) are applied in the fabrication and testing of an optical system with an annular pupil, the difference function represents the polishing error due to the use of the circle polynomials.

If we compare the annular coefficients of astigmatism, coma, and spherical aberration given by Eqs. (39d)–(39f) with the corresponding Zernike coefficients given by Eq. (46d)–(46f), we obtain

$$\frac{a_6}{\hat{b}_6} = (1+\epsilon^2+\epsilon^4)^{1/2}, \tag{50a}$$

$$\frac{a_8}{\hat{b}_8} = (1-\epsilon^2)\left(\frac{1+4\epsilon^2+\epsilon^4}{1+\epsilon^2}\right)^{1/2}, \tag{50b}$$

$$\frac{a_{11}}{\hat{b}_{11}} = (1-\epsilon^2)^2. \tag{50c}$$

Because the $\hat{b}_j$ coefficients are independent of the value of ϵ, the variation of a ratio $a_j/\hat{b}_j$ with ϵ represents the variation of an annular coefficient a_j.

Now we consider the expansion of the Seidel aberration function in terms of the circle polynomials by assuming them to be orthogonal over the annulus. This is what one does when defining a center of an interferogram, drawing a unit circle around it, and determining its circle coefficients. The aberration function in this case can be written in the form

$$W(\rho,\theta;\epsilon) = b_1Z_1 + b_2Z_2 + b_4Z_4 + b_6Z_6 + b_8Z_8 + b_{11}Z_{11} + \ldots, \tag{51}$$

where, according to Eq. (17), the coefficients b_j are given by

$$b_j = \frac{1}{\pi(1-\epsilon^2)}\int_\epsilon^1\int_0^{2\pi} W(\rho,\theta;\epsilon)Z_j(\rho,\theta)\rho\mathrm{d}\rho\mathrm{d}\theta. \tag{52}$$

They can also be obtained from Eq. (22), i.e., from the annular or circle coefficients by using the matrix $\boldsymbol{C}^{ZZ}$ or $\boldsymbol{C}^{ZF}$ given in Tables 4 and 5, respectively. The "incorrect" circle coefficients b_j are given by

$$b_1 = a_1, \tag{53a}$$

$$b_2 = (1+\epsilon^2)^{1/2}a_2, \tag{53b}$$

$$b_4 = \frac{1}{4\sqrt{3}}(1+\epsilon^2+4\epsilon^4)(2A_d+A_a) + \frac{1}{2\sqrt{3}}(1+\epsilon^2+\epsilon^4+3\epsilon^6)A_s, \tag{53c}$$

$$b_6 = (1+\epsilon^2+\epsilon^4)^{1/2}a_6, \tag{53d}$$

$$b_8 = \sqrt{2}\epsilon^4A_t + \frac{1}{6\sqrt{2}}(1+\epsilon^2+\epsilon^4+9\epsilon^6)A_c, \tag{53e}$$

$$b_{11} = \frac{\sqrt{5}}{4}\epsilon^4(3\epsilon^2-1)(2A_d+A_a) + \frac{1}{6\sqrt{5}}(1+\epsilon^2+\epsilon^4-9\epsilon^6+36\epsilon^8)A_s, \tag{53f}$$

etc. These coefficients are incorrect in the sense that they do not yield a least-squares fit of the aberration function. Because an annular polynomial with $n = m$ has the same form as that for a corresponding circle polynomial except for the normalization constant, the coefficients b_j and a_j for such a polynomial are also related to each other by the normalization constant. Equations (53a), (53b), and (53d) represent this fact for $n = m = 0$, 1, 2, respectively. It is clear, however, that the improperly calculated circle coefficients b_j depend on the obscuration ratio of the pupil. Evidently, they are different from the corresponding $\hat{b}$ coefficients given by Eqs. (46). While the value of the piston coefficient b_1 is equal to the true mean value a_1, the tilt coefficient b_2 is larger than a_2 by a factor of $(1+\epsilon^2)^{1/2}$ or 1.1180, and the coma coefficient b_6 is larger than a_6 by a factor of

$$(1+\epsilon^2+\epsilon^4)^{1/2}$$

or 1.1456 when $\epsilon = 0.5$. Moreover, the b coefficients of some of the nonexistent higher-order aberrations are not zero. For example, the coefficients b_{22}, b_{37}, etc., of the secondary and tertiary Zernike spherical aberrations Z_{22}, Z_{37}, etc., and b_{16}, b_{30}, etc., of the secondary and tertiary Zernike coma Z_{16} and Z_{30}, etc., are nonzero. Thus, nonexistent aberrations are generated when an aberration function is expanded improperly in terms of the circle polynomials.

If we estimate the annular Seidel aberration function with only four-circle polynomials from Eq. (51), we obtain

$$\hat{W}(\rho,\theta;\epsilon) = b_1Z_1 + b_2Z_2 + b_4Z_4. \tag{54}$$

If we truncate the expansion in terms of the circle polynomials in Eq. (51) to the first 11 circle polynomials and remove the first four coefficients as interferometer setting errors, the residual aberration function in this case is given by

$$W_{\mathrm{RC}b}(\rho,\theta;\epsilon) = b_6Z_6 + b_8Z_8 + b_{11}Z_{11}. \tag{55}$$

The tilt error is larger by a factor of $(1+\epsilon^2)^{1/2}$ or 1.1180, when $\epsilon = 0.5$, than its true value given by a_2, and the defocus error given by b_4 can be compared with its true value given by a_4. Because the 11-polynomial aberration function from Eq. (51) is not equal to the aberration function of Eq. (41), their difference does not consist of the difference in their interferometer setting errors. For example, Eq. (53d) indicates that there will be an astigmatism term in the difference function. Thus, wrong polishing will result if the aberration function of Eq. (55) is provided to the optician to zero out.

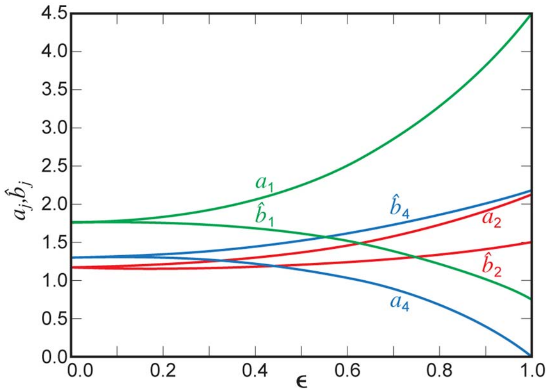

Fig. 1. (Color online) Orthonormal annular coefficients a_j and Zernike circle coefficients $\hat{b}_j$ for a four-polynomial expansion.

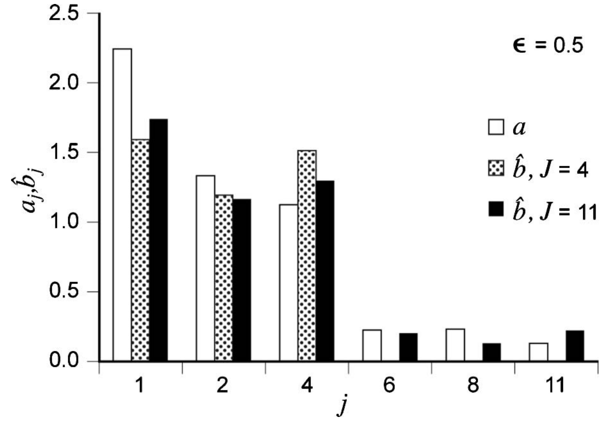

Fig. 3. Orthonormal annular coefficients a_j and Zernike circle coefficients $\hat{b}_j$, illustrating how the latter change as the number of polynomials changes from 4 to 11.

As a numerical example, we consider an annular Seidel aberration function with $A_t = A_d = A_a = 1$, $A_c = 2$, and $A_s = 3$ in waves. As illustrated in Fig. 1, the annular and circle coefficients of a four-polynomial expansion differ from each other, although they yield the same fit of the aberration function. We note that, whereas the mean value a_1 increases as ϵ increases, the piston coefficient $\hat{b}_1$ decreases. However, the defocus coefficient a_4 decreases, but $\hat{b}_4$ increases. Both tilt coefficients a_2 and $\hat{b}_2$ increase. For an 11-polynomial expansion, the first four annular coefficients remain the same, but the circle coefficients become independent of ϵ, as in Eqs. (46). Figure 2 shows the coefficient ratios $a_6/\hat{b}_6$ (astigmatism), $a_8/\hat{b}_8$ (coma), and $a_{11}/\hat{b}_{11}$ (spherical) for an 11-polynomial expansion. We note that the coefficient a_6 increases, a_{11} decreases, and a_8 is nearly constant for small values of ϵ and then decreases as ϵ increases. Figure 3 shows how the $\hat{b}$ coefficients change as we change the number of polynomials from 4 to 11 for $\epsilon = 0.5$. A wrong polishing will result if the tip, tilt, and focus errors of an interferometer setting are estimated from the 11-circle-polynomial expansion, instead of the four. The variation of standard deviation obtained from the coefficients of a 4- or 11-polynomial expansion is shown in Fig. 4, illustrating that the circle coefficients yield incorrect results. The standard deviation obtained from the orthonormal coefficients increases slowly with ϵ, starting at 1.7460 and 1.7877 for the 4- and 11-polynomial expansions, respectively. However, the standard deviation obtained from the circle coefficients is correct only when $\epsilon = 0$. It increases rapidly with ϵ for the 4-polynomial expansion, but it is constant for the 11-polynomial expansion, indicating its incorrect nature. The sigma values from the orthonormal and the circle coefficients are nearly equal to each other for $\epsilon \leq 0.5$ because of the very slow increase of the orthonormal sigma.

Figure 5 shows the contours of the Seidel aberration function for a circular and an annular pupil with obscuration ratio of $\epsilon = 0.5$. The case of a circular pupil is included just for reference. The dark circular region in Fig. 5(b) (and others) represents the obscuration. The contours of the annular Seidel aberration function fit with only four polynomials, as in Eq. (38) or (43) and in Eq. (54), are shown in Figs. 6(a) and 6(b), respectively. The two figures look similar, but are not the same. Only Fig. 6(a) represents the least-squares and, therefore, the correct fit. The contours of the residual

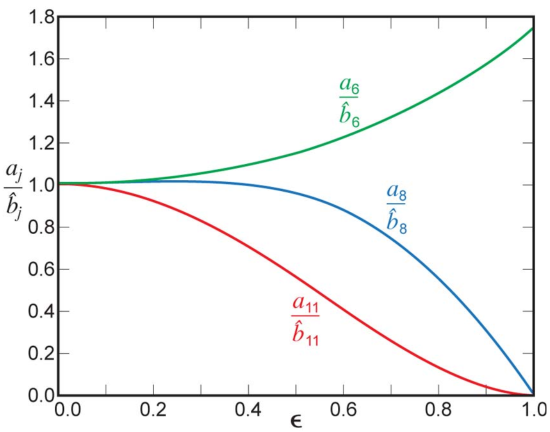

Fig. 2. (Color online) Ratio of the orthonormal annular coefficients a_j and Zernike circle coefficients $\hat{b}_j$ for an 11-polynomial expansion.

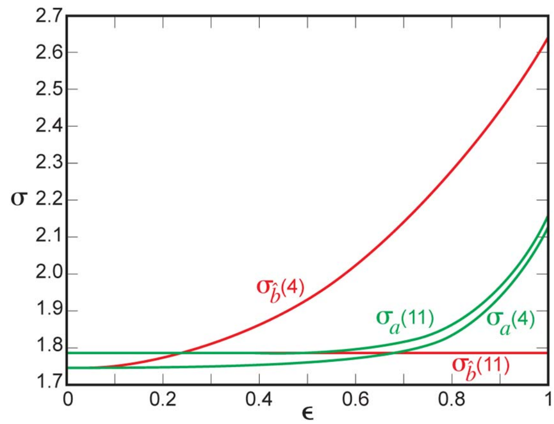

Fig. 4. (Color online) Standard deviation as obtained from the orthonormal annular coefficients a_j and Zernike circle coefficients $\hat{b}_j$ of a 4- and 11-polynomial expansion.

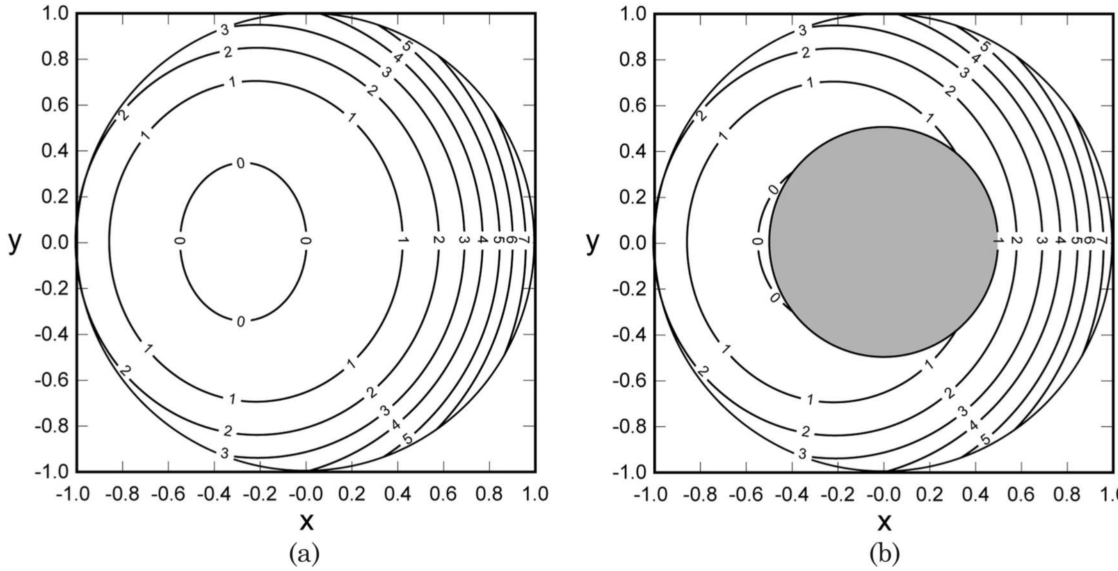

Fig. 5. Contours of (a) Seidel aberration function of Eq. (37) for a circular pupil with $A_t = A_d = A_a = 1$, $A_c = 2$, and $A_s = 3$ in waves. (b) Same Seidel aberration function, but for an annular pupil with obscuration ratio $\epsilon = 0.5$.

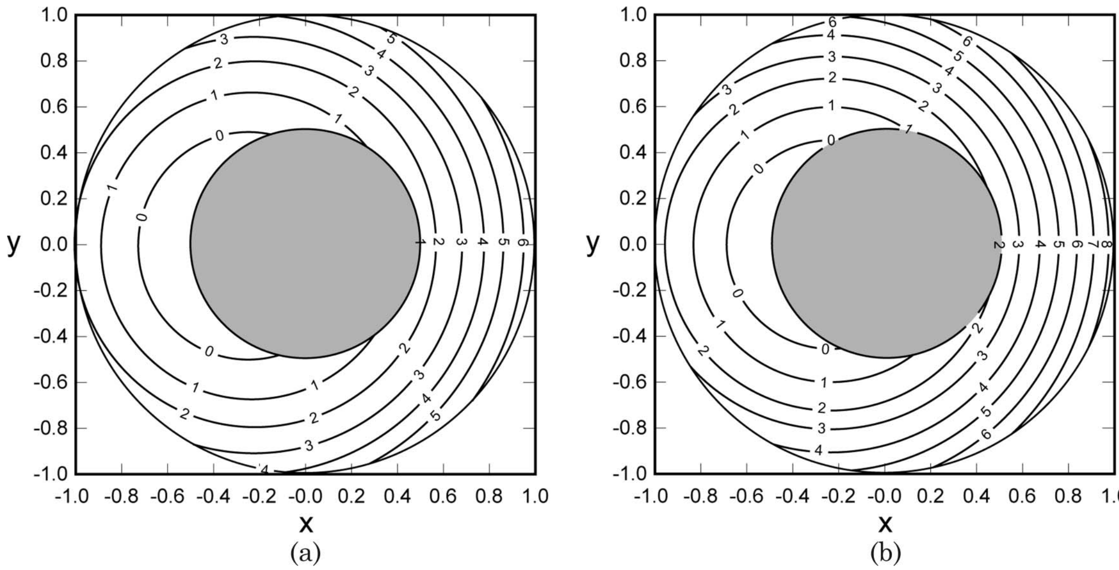

Fig. 6. Contours of an annular Seidel aberration function for $\epsilon = 0.5$ fit with only four polynomials, as in (a) Eq. (38) or (43), and (b) Eq. (54).

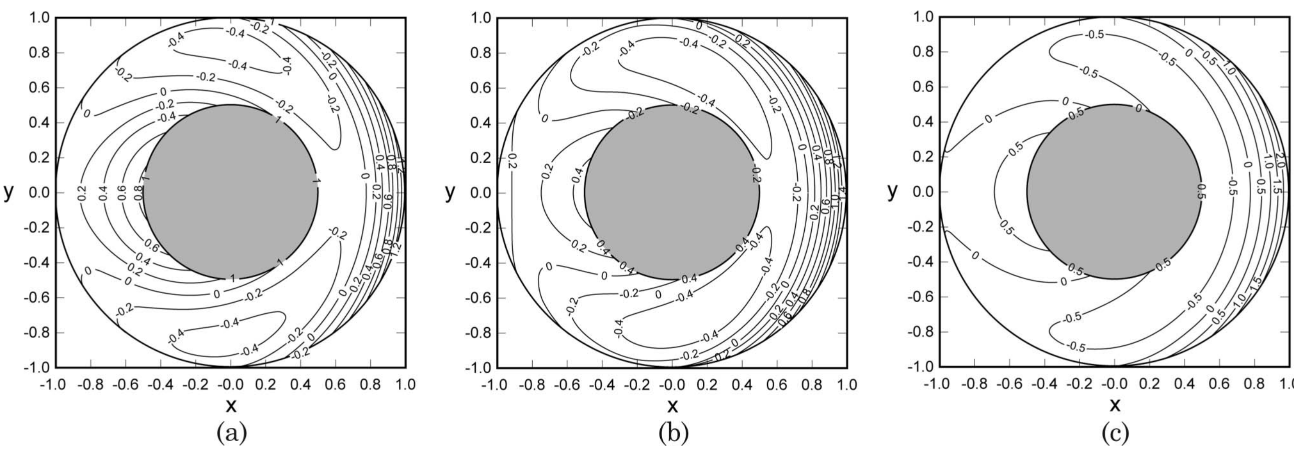

Fig. 7. Contours of the residual aberration function after removing the interferometer setting errors. (a) W_{RA} of Eq. (47) using annular polynomials (b) $W_{\mathrm{RC}\hat{b}}$ of Eq. (48) using circle polynomials correctly, and (c) $W_{\mathrm{RC}b}$ of Eq. (55) using circle polynomials incorrectly.

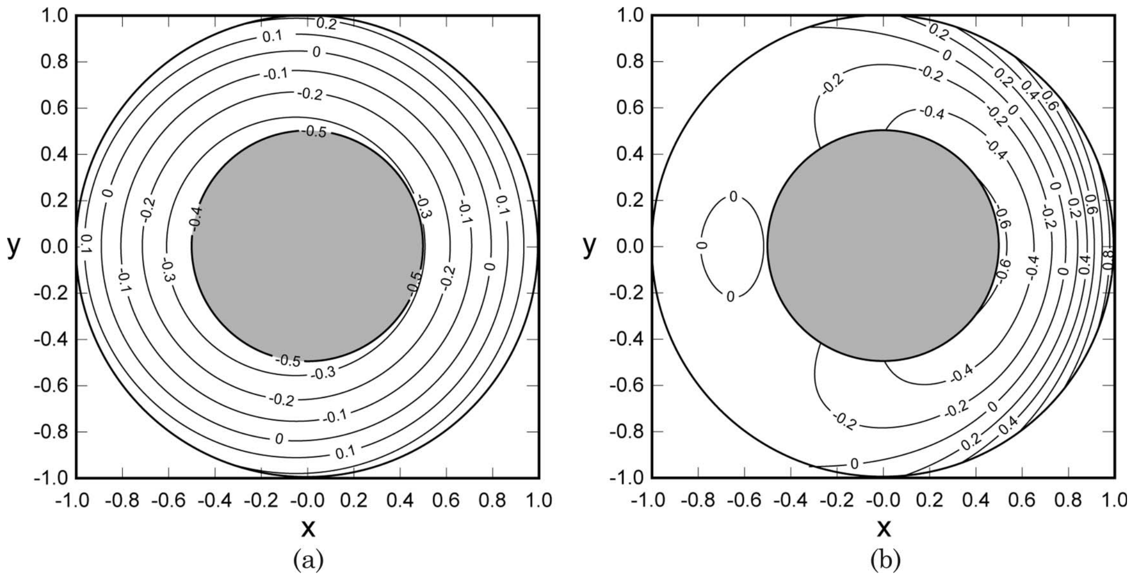

Fig. 8. Contours of the difference or the error function (a) Eq. (49) and (b) obtained by subtracting Eq. (47) from Eq. (55).

aberration function when the first four (of the 11) polynomials are removed as interferometer setting errors, as in Eqs. (47), (48), and (55), are shown in Figs. 7(a)–7(c), respectively. All of the three figures are different from each other, as expected. Only Fig. 7(a) reflects removal of the correct interferometer setting errors, and thus the correct residual aberration function. The contours of the difference of the residual functions using the circle polynomials from the one using the annular polynomials given by Eq. (49) and by subtracting Eq. (47) from Eq. (55) are shown in Figs. 8(a) and 8(b). They represent the error functions due to removal of incorrect interferometer setting errors.

8. Aberration Coefficients From Discrete Wavefront Data

When an aberration function is known only at a discrete set of points, as in a digitized interferogram, the integral for determining the aberration coefficients reduces to a sum and the orthonormal coefficients thus obtained may be in error due to the lack of orthogonality of the polynomials over the discrete points of the aberration data set. The magnitude of the error decreases as the number of points distributed uniformly across an interferogram increases. This is not a serious problem when the wavefront errors are determined by, say, the phase-shifting interferometry, since the number of points can be very large [10]. However, when the number of data points is small, or the pupil is irregular in shape due to vignetting, then ray tracing or testing of the system yields wavefront error data at an array of points across a region for which the closed-form orthonormal polynomials are not available. In such cases, we can determine the coefficients of an expansion in terms of the numerical polynomials that are orthogonal over the data set, obtained by the Gram–Schmidt orthogonalization process [11]. However, if we just want to determine the values of tip/tilt and defocus terms, yielding the errors in the interferometer settings, they can be obtained by least-squares fitting the aberration function data with only these terms.

9. Discussion and Conclusions

After a brief review of the theory of expansion of a noncircular aberration function in terms of the polynomials that are orthonormal over the domain of the function and in terms of the Zernike circle polynomials, an annular wavefront is considered. It is shown that, whereas the orthonormal annular expansion coefficients are independent of the number of polynomials used in the expansion, the circle coefficients generally change as the number of polynomials changes. In fact, it is easy to see which coefficients change and by how much. Accordingly, one or more annular polynomials can be added to or subtracted from the aberration function without affecting the coefficients of the other polynomials. Moreover, unlike the annular coefficients, the piston circle coefficient does not represent the mean value of the aberration function, and the sum of the squares of the other coefficients does not yield its variance. However, since each annular polynomial of a certain order is a linear combination of the circle polynomials of that and lower orders, the wavefront fit with a certain number of annular polynomials is identically the same as that with the corresponding circle polynomials. Accordingly, the interferometer setting errors of tip, tilt, and defocus are the same with a four-circle-polynomial expansion as those from the annular-polynomial expansion. However, incorrect setting errors are obtained when, for example, obtained from the corresponding coefficients of an 11-circle-polynomial expansion. Consequently, when these errors are removed from the aberration function, wrong polishing will result by zeroing out the residual aberration function.

These results are illustrated analytically as well as numerically by considering an annular Seidel aberration function. Both the 4- and 11-polynomial expansions in terms of the annular and circle polynomials are considered, and their differences are discussed. It is shown, for example, that whereas the nonzero annular coefficients of an 11-polynomial expansion depend on the obscuration ratio, the circle coefficients are independent of it. While an expansion in terms of any number of annular polynomials (larger than the first four) yields the correct interferometer setting errors, only a four-polynomial expansion in terms of the circle polynomials yields their correct values.

For an annular wavefront shown in Fig. 5(b), Figs. 6(a) and 7(a) represent the correct 4-polynomial least-squares fit and the corresponding correct residual aberration function. The same results are obtained whether the annular or the circle polynomials are used in the fitting process. However, if 11 polynomials are used to estimate the aberration function and the first four are removed as interferometer setting errors, then only the annular polynomials give the correct residual aberration function. The residual aberration functions shown in Figs. 7(b) and 7(c) obtained respectively by using the circle polynomials in a least-squares fit or assuming their orthogonality over the annulus, are incorrect. The error functions representing the difference between the correct and the incorrect residual aberration functions are shown in Figs. 8(a) and 8(b). These figures illustrate that, while the correct interferometer setting errors can be obtained by a 4-polynomial least-squares fit using the annular or the circle polynomials, only the annular polynomials yield their correct values when obtained by fitting with a larger number of polynomials.

If the common practice of defining the center of an interferogram and drawing a unit circle around it is followed, then the circle coefficients of a noncircular interferogram do not yield a correct representation of the aberration function. Moreover, in this case, some of the higher-order coefficients of aberrations that are nonexistent in the aberration function are also nonzero. Finally, the circle coefficients, however obtained, do not represent the coefficients of the balanced aberrations for an annular pupil. Consequently, it should be clear that the circle polynomials are not suitable for the analysis of an annular wavefront, and only the annular polynomials should be used for such an analysis.

Although we have considered annular wavefronts as an example of a noncircular wavefront, the results obtained and illustrated for them are also applicable to other noncircular wavefronts, such as hexagonal or square.

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